Universidade de Aveiro Departamento de Matemática
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Natália da Costa Martins

Análise Não-Standard em Equações Diferenciais Ordinárias e na Teoria dos Pontos Críticos

Nonstandard Analysis in Ordinary Differential Equations and in Critical Point Theory

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Vítor Manuel Carvalho das Neves, Professor Associado do Departamento de Matemática da Universidade de Aveiro e co-orientação de Maria João Simões Nunes Borges, Professora Auxiliar do Departamento de Matemática do Instituto Superior Técnico da Universidade Técnica de Lisboa.

Dissertation submitted to the University of Aveiro in fulfilment of the requirements for the degree of *Doctor of Philosophy* in Mathematics, under the supervision of Vítor Manuel Carvalho das Neves, Associated Professor at the Department of Mathematics of the University of Aveiro and co-supervision of Maria João Simões Nunes Borges, Assistant Professor at the Department of Mathematics of the *Instituto Superior Técnico* of the Technical University of Lisbon.

o júri

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palavras-chave Análise Não-Standard, Teoria Abstrata dos Pontos Críticos, Problemas de Minimax, Equações Diferenciais Ordinárias.

resumo

Esta dissertação apresenta várias aplicações da Análise Não-Standard à Teoria das Equações Diferenciais Ordinárias e à Teoria dos Pontos Críticos.

Relativamente à Teoria das Equações Diferenciais Ordinárias, são apresentadas generalizações não-standard de dois resultados importantes desta teoria, bem como uma nova prova não-standard do Teorema de Existência de Carathéodory e dedução correspondente do Teorema de Existência de Peano.

Um dos resultados fundamentais da Teoria dos Pontos Críticos é o Teorema da Passagem da Montanha de Ambrosetti-Rabinowitz. Neste contexto, são apresentadas várias provas não-standard deste teorema para funcionais coercivos definidos em espaços de Banach reais de dimensão finita, além de várias generalizações não-standard de condições do tipo de Palais-Smale que permitem a demonstração de novos teoremas. São ainda apresentados dois novos teoremas da passagem da montanha sem a condição de Palais-Smale ou suas generalizações. Todos estes teoremas permitem obter novos *teoremas de três pontos críticos*.

keywords

Nonstandard Analysis, Abstract Critical Point Theory, Minimax Problems, Ordinary Differential Equations.

abstract

This dissertation describes several applications of Nonstandard Analysis both to the Ordinary Differential Equations Theory and to the Critical Point Theory.

Two important results of Ordinary Differential Equations Theory are generalized according to Nonstandard Analysis, a new nonstandard proof of Carathéodory's Existence Theorem is presented wherefrom Peano's Existence Theorem is deduced.

One of the fundamental results of Critical Point Theory is the Mountain Pass Theorem of Ambrosetti-Rabinowitz. Several nonstandard proofs of this theorem for coercive functionals defined in finite dimensional real Banach spaces are presented together with some nonstandard generalizations of Palais-Smale conditions that allow the demonstration of new theorems. Two new *mountain pass theorems* are also proved without using the Palais-Smale condition or generalizations thereof. These *mountain pass theorems* are used to obtain new *three critical points theorems*.

Ao Quim,
à Inês, ao Nuno e ao Miguel.

Contents

Symbols

Basic notation:

$:=$	equality, by definition;
$\emptyset$	empty set;
$\mathbb{N}$	set of natural numbers;
$\mathbb{Z}$	set of integer numbers;
$\mathbb{Q}$	set of rational numbers;
$\mathbb{R}$	set of real numbers;
$\mathbb{R}^+$	set of positive real numbers;
$\mathbb{R}_0^+$	set of nonnegative real numbers;
$\mathbb{R}^n$	n-dimensional Euclidean space;
$\mathbf{B}_r(a)$	open ball centered at a and radius $r \in \mathbb{R}^+$;
$\overline{\mathbf{B}}_r(a)$	closed ball centered at a and radius $r \in \mathbb{R}^+$;
$card(E)$	cardinality of the set E;
$\mathcal{P}(E)$	power set of E;
B^A	set of all mappings from the set A into the set B;
$\dot{\bigcup}$	disjoint union;

$f_{\|A}$	restriction of the map $f : X \to Y$ to $A \subseteq X$;
$\cdot$	takes the place of the variable with respect to which the mapping is evaluated;
$\|\cdot\|$	absolute value function;
$\\|\cdot\\|$	norm;
$\cdot \bullet \cdot$	inner product;
$\blacksquare$	end of proof.

Other symbols and pages where they are introduced:

Chapter 1

Introduction

1.1 Overview

Nonstandard Analysis (**NSA**) is a coherent and powerful theory developed by Abraham Robinson in 1961 [Rob61], which among other things, provides a logical foundation for the use of infinitesimal numbers in Mathematics. Robinson proved that the set of real numbers, $\mathbb{R}$, may be made a proper subset of a new set of numbers, which will thus contain infinitesimal numbers and infinite numbers and in a sense satisfies the same analytical properties of $\mathbb{R}$; namely, functions extend naturally with the same properties. This new set is usually denoted by ${}^\star\mathbb{R}$ and its elements are called hyperreal numbers or nonstandard real numbers. **NSA** is also sometimes referred in the literature to as Infinitesimal Analysis or Robinsonian Analysis.

One of the advantages of **NSA** is that nonstandard methods may yield shorter and/or intuitive proofs of classical results. However, nonstandard methods are more than an alternative to standard methods, since they have provided new results in many fields of Mathematics, such as Functionals Analysis, Differential Equations, Optimal Control Theory, Probability Theory and Mathematical Physics. Even more, **NSA** is an important field of research in their own right; [ACH97] and [CNOSP95] are collection of articles covering basic **NSA** as well as advanced material from many of the areas of Mathematics to which **NSA** has being applied.

Critical Point Theory (**CPT**) is also an important subject in Mathematics that has been widely used in the last decades in many fields of Mathematics, such as Differential Equations, Differential Geometry and Global Analysis and also in Physics and Mechanics, where solutions of many problems are critical points of a suitable energy functional.

CPT rely on methods of classical Mathematics. However, we believe that the application of nonstandard methods to **CPT** may both simplify and potentiate the development of new results.

According to [San93], this theory has its origins in the Calculus of Variations and it had an increased development after the first quarter of the twentieth century with the works of Morse, Lusternik and Schnirelmann. In contrast to the Calculus of Variations, where the problems involve the determination of maxima and minima of functionals, **CPT** concerns the study of all types of critical points.

Until the second half of the nineteenth century, the existence of a minimum was taken for granted if the functional was bounded from below. This situation changed in 1870 when Weierstrass [Wei] gave an example of a nonnegative functional which did not have a minimum. The basic idea for finding a minimum of a smooth functional is, in general, simple: if E is a real Banach space and $f : E \to \mathbb{R}$ is of class C^1 and bounded from below, with a *deformation technique* or using Ekeland's Variational Principle [Eke79], we can construct a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$f(u_n) \to \inf_{t\in E} f(t) \quad \text{and} \quad f'(u_n) \to 0.$$

The main problem is to show that this sequence has an accumulation point which will be a minimizer. If f satisfies the condition

> *every sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $(f(x_n))_{n\in\mathbb{N}}$ is bounded and $f'(x_n) \to 0$ has a convergent subsequence,*

the functional has a minimum. This condition was introduced in 1964 by Palais and Smale [PS64] and it is known by Palais-Smale condition ((**PS**) for short). In the beginning, this condition was received with some caution because several interesting functionals did not satisfy it. Since then, many variants of the Palais-Smale condition have been introduced and nowadays these Palais-Smale conditions became important *compactness conditions* in many critical point theorems. In 1980, H. Brézis, J. Coron and L. Nirenberg introduced the following generalization of the Palais-Smale condition [BCN80]:

> *every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* E *such that* $f(x_n) \to c$ *and* $f'(x_n) \to 0$ *has a convergent subsequence,*

where c is a fixed real number. This Palais-Smale condition of level c ($(\mathbf{PS})_c$ for short) is a *compactness condition* on the functional f in the sense that the set of critical points of f with value c,

$$K_c := \{u \in E : f'(u) = 0 \ \wedge \ f(u) = c\},$$

is compact.

Saddle points, that is, critical points that are not local extrema, are more difficult to find but lead to similar compactness problems. *Deformation techniques* were introduced in 1934 by Lusternik and Schnirelman [LS34] and can also be applied to functionals f which do not need to be bounded from below (or above), by characterizing the critical values as a *minimax* over a suitable class of sets $\mathcal{S}$:

$$\inf_{A\in\mathcal{S}} \sup_{x\in A} f(x) := c.$$

The choice of $\mathcal{S}$ must reflect some change in the topology of the sublevel sets

$$f^s := \{u \in E : f(u) \le s\}$$

for s near c.

In the general case, *minimax theorems* for C^1 functionals are obtained as follows:

1. the functional f must satisfy some *geometrical condition* that relates the value of the functional over some sets that have some kind of connection between them;

2. use a *deformation technique* to prove that, for some value c characterized by a *minimax argument*, there exists a Palais-Smale sequence of level c, that is, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $f(x_n) \to c$ and $f'(x_n) \to 0$;

3. use a Palais-Smale type *compactness condition* to prove that c is a critical value of f.

In this work we will present the Quantitative Deformation Lemma for C^1 functionals, introduced in 1983 by Willem [Wil83], as an example of a *deformation technique*. We mention that there also exists Deformation Lemmas for continuous functionals defined on metric spaces (see, for example, [Cor99]) and even for non continuous functionals (see, [RTK98]). These results are fundamental for the development of nonsmooth critical point theory.

An important *minimax theorem* that will be addressed in this dissertation is the Mountain Pass Theorem [AR73] introduced in 1973 by Ambrosetti and Rabinowitz. This theorem considers a C^1 functional f defined in a real Banach space E that verifies both the $(\mathbf{PS})$ condition and the following condition

$$\textit{there exist } x_1, x_2 \in E \textit{ and } r \in \mathbb{R}^+ \textit{ such that } \| x_1 - x_2 \| > r \textit{ and} \qquad (*)$$

$$k_0 := \max\{f(x_1), f(x_2)\} \; < \inf_{\|y - x_1\| = r} f(y).$$

The Mountain Pass Theorem of Ambrosetti-Rabinowitz shows that there exists a critical point u different from x_1 and x_2 and with critical value $k_1 > k_0$. Moreover, k_1 is the following *minimax value*

$$k_1 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$$

where Γ is the set of all continuous paths joining x_1 to x_2.

Let us give a geometric interpretation of this theorem; if $E = \mathbb{R}^2$ and, for each x, $f(x)$ represents the altitude of x, then x_1 and x_2 are two different points that are separated by a mountain range. For each $\gamma \in \Gamma$, the number $\max_{t \in [0,1]} f(\gamma(t))$ is the maximum height on that path and k_1 is the infimum of all those maximal heights. In order to go from x_1 to x_2 one tries to find a mountain pass $\gamma_0 \in \Gamma$ such that

$$\max_{t \in [0,1]} f(\gamma_0(t)) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)) = k_1.$$

The geometry expressed by condition (*) is called the mountain pass geometry and is the simplest *minimax geometry* that leads to a *minimax theorem*. Using the Quantitative Deformation Lemma and the mountain pass geometry, one can prove that there exists a sequence $(u_n)_{n \in \mathbb{N}}$ such that

$$f(u_n) \to k_1 \quad \text{and} \quad f'(u_n) \to 0.$$

Therefore, since f satisfies $(\mathbf{PS})$, then there exists a critical point with value k_1. Notice that only the condition $(\mathbf{PS})_{k_1}$ is needed to conclude that k_1 is a critical value as it was pointed out by Brézis, Coron and Nirenberg in [BCN80], the paper where the condition $(\mathbf{PS})_{k_1}$ was introduced for the first time. This generalization of the Mountain Pass Theorem of Ambrosetti-Rabinowitz is known as the Mountain Pass Theorem of Brézis-Coron-Nirenberg.

Note that if f does not satisfy $(\mathbf{PS})_{k_1}$, we cannot guarantee that the smallest value of $\max_{t\in[0,1]} f(\gamma(t))$ exists (see Exemple 5.13).

We notice that the critical point obtained by the Mountain Pass Theorem of Ambrosetti-Rabinowitz (and by the Mountain Pass Theorem of Brézis-Coron-Nirenberg) is not necessarily a saddle point, as one might conjecture from the mountain pass geometry. Pucci and Serrin showed in [PS84] that under certain reasonable hypotheses, the critical point must indeed be a saddle point. In particular, they proved that if there is only one critical point with the critical value k_1, then it is a saddle point.

From the Mountain Pass Theorem of Ambrosetti-Rabinowitz it follows easily that if f is a C^1 functional that satisfies (**PS**) and has two distinct local minimizers, then f has a third critical point. This result is known as a Three Critical Points Theorem.

Research activities around Mountain Pass Theorem of Ambrosetti-Rabinowitz have produced a great variety of generalizations of this theorem. Such generalizations were obtained using weaker Palais-Smale conditions, weakening the differentiability of the functional and by adopting more general geometrical conditions than the mountain pass geometry.

Readers interested in the Mountain Pass Theorem of Ambrosetti-Rabinowitz and its generalizations, may refer to [GT01] and [Jab03] for further details.

1.2 Contributions

This dissertation applies for the first time, as far as we know, Nonstandard Analysis to Critical Point Theory. It also presents a new and general form to prove some important theorems for ordinary differential equations (**ODE's**).

The major contributions of this dissertation are summarized next.

1.2.1 Nonstandard Carathéodory's Existence Theorem

A nonstandard proof of Carathéodory's Existence Theorem (which avoid Ascoli's Theorem as well as Lebesgue's Dominated Convergence Theorem) is presented wherefrom Peano's Existence Theorem is deduced. For this purpose, we use the Loeb integration theory and the notion of nonstandard discrete derivative. We also obtained a generalization of Carathéodory's Existence Theorem, that we called Nonstandard Carathéodory's Existence Theorem (Theorem 3.1).

1.2.2 Nonstandard Palais-Smale conditions

Nonstandard conditions of Palais-Smale type are presented. Inspired in (**PS**) and $(\mathbf{PS})_c$ conditions, we obtained several nonstandard variants of these classical conditions that are weaker than the classical ones, but still sufficient to prove new Mountain Pass Theorems.

1.2.3 Theorems with nonstandard Palais-Smale conditions

We establish some relations between coercivity and our nonstandard Palais-Smale conditions (Proposition 4.24 and Proposition 5.8). We also obtained a *minimizing theorem* which is a generalization of a classical result (Corollary 5.4).

1.2.4 Mountain Pass Theorem with a nonstandard Palais-Smale condition

The nonstandard Palais-Smale conditions *per* level allow us to prove a Mountain Pass Theorem which is a generalization of the Mountain Pass Theorem of Brézis-Coron-Nirenberg (Theorem 5.15). We also present the "dual" of this new Mountain Pass Theorem (Theorem 5.18).

1.2.5 Mountain Pass Theorems without Palais-Smale conditions

We proved that, in the finite dimensional case, if we substitute the (**PS**) condition in the Mountain Pass Theorem of Ambrosetti-Rabinowitz by *there exists* $s \in \mathbb{R}^+$ *such that* $\| x_2 - x_1 \| < s$ *and if* $\| x - x_1 \| \geq s$ *then* $f(x) < k_1$

we obtain other theorem of mountain pass type (Theorem 5.25).

We also proved a new Mountain Pass Theorem without Palais-Smale conditions (Theorem 5.29) for a functional f defined in a real Hilbert space H that satisfies the condition

$$\exists \gamma \in {}^\star\Gamma \; [\; \gamma({}^\star[0,1]) \subseteq ns({}^\star H) \;\wedge\; \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1 \;].$$

Notice that these theorems cannot be obtained from the Mountain Pass Theorem of Ambrosetti-Rabinowitz or generalizations of it, e.g. given in [GT01], since our geometrical conditions do not imply (**PS**) or weaker forms of (**PS**). We also present the "duals" of these new Mountain Pass Theorems (Theorem 5.28 and Theorem 5.31).

1.2.6 Nonstandard proofs of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for coercive functionals defined in finite dimensional real Banach spaces

We present some nonstandard proofs of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for coercive functionals defined in finite dimensional real Banach spaces without using a Deformation Lemma. One of the proofs uses a hyperfinite set and is of a discrete type.

1.2.7 Three Critical Points Theorems with a nonstandard Palais-Smale condition

From the Mountain Pass Theorem with nonstandard Palais-Smale condition and its "dual", we obtained new Three Critical Points Theorems (Theorem 6.1, Theorem 6.2, Theorem 6.4 and Theorem 6.5).

1.2.8 Three Critical Points Theorems without Palais-Smale conditions

Applying the Mountain Pass Theorems without Palais-Smale conditions and their "duals", we proved other Three Critical Points Theorems (Theorem 6.7 to Theorem 6.10, Theorem 6.12

to Theorem 6.15).

1.3 Organization of the dissertation

We had the concern to guide the exposition in a logical and sequential way. In order to create a self contained text, we present the contents that we found necessary for the understanding of this work.

This dissertation is organized in six chapters and two appendixes. The Index, Symbols and Bibliography refer the reader to terminology, notation and sources thereof.

In **Chapter 2** we present a brief introduction to Nonstandard Analysis. We begin this chapter presenting the fundamental results for our almost axiomatic description of Nonstandard Analysis: the Transfer Principle and the Polysaturation Principle. Then, in the setting of real normed spaces, we present nonstandard characterizations of some topological concepts. We also present some nonstandard characterizations of C^1 functions defined in real Banach spaces. A great part of this chapter is dedicated to Loeb integration theory and the relations between this theory and the Lebesgue integration theory. At the end of this chapter, we introduce the notion of nonstandard discrete derivative.

In **Chapter 3** we present a nonstandard generalization of Carathéodory's Existence Theorem and a nonstandard proof of Carathéodory's Existence Theorem. In the end of this chapter, we obtain Peano's Existence Theorem as a corollary of Carathéodory's Existence Theorem.

We begin **Chapter 4** presenting the classical (**PS**) condition and our nonstandard variants of (**PS**). After these definitions, we establish some relations between these nonstandard conditions and (**PS**). In order to relate coercivity and our nonstandard Palais-Smale conditions, we present some nonstandard characterizations of coercive functionals. To end this chapter, we present $(\mathbf{PS})_c$ condition, our nonstandard variants of $(\mathbf{PS})_c$ and the relations between them.

Chapter 5 contains a (known) proof of a variant of Ekeland's Variational Principle using the Quantitative Deformation Lemma. This variational principle and a nonstandard Palais-Smale condition proves a generalization of a classical minimizing theorem. We proceed by presenting the famous Mountain Pass Theorem of Ambrosetti-Rabinowitz and one of its generalizations, the Mountain Pass Theorem of Brézis-Coron-Nirenberg. Using the Quantitative Deformation Lemma and a nonstandard Palais-Smale condition, we prove a new generalization of the Mountain Pass Theorem of Brézis-Coron-Nirenberg. Afterwards, we present a proof of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for coercive functionals defined in finite dimensional real Banach spaces without using the Quantitative Deformation Lemma. In the end of this chapter, we prove two new Mountain Pass Theorems without Palais-Smale conditions.

In **Chapter 6** we prove some variants of Three Critical Points Theorems: Three Critical Points Theorems without Palais-Smale conditions and Three Critical Points Theorems with a nonstandard Palais-Smale condition.

In **Appendix A** we present a nonstandard generalization of Peano's Existence Theorem and a (known) nonstandard proof of Peano's Existence Theorem.

In **Appendix B** we present other two nonstandard proofs of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for coercive functionals defined in finite dimensional real Banach spaces.

Chapter 2

Nonstandard Analysis

2.1 Introduction

The aim of this chapter is to present a short introduction to Nonstandard Analysis. In Section 2.2 and Section 2.3 we will present basic concepts and results about **NSA**. In Sections 2.4 and 2.5 we will see how **NSA** makes many mathematical arguments and concepts much easier than the classical ones. For example, we will see that the nonstandard characterizations of C^1 functions (see Theorems 2.37 and 2.38) are much simpler than the classical ones and easier to work with.

In Section 2.6 we will describe Loeb's integration theory and its relation with Lebesgue's integration theory. We will see that the Lebesgue integral is *infinitely close* to an appropriate *hyperfinite sum* (see Theorem 2.66). Finally, in Section 2.7, we will present the notion of nonstandard discrete derivative and some of its properties.

In this introduction to **NSA** we will omit all the proofs and many technical details, because we do not intend to make a detailed presentation of foundations of **NSA**, but only to present the notions and results needed in the next chapters, in order to keep the work more self contained; specified references are sources for proofs.

2.2 Notation and preliminars in NSA

In this section we will give an almost axiomatic description of the foundations of **NSA**. For other approaches the reader may consult the references [Lin88], [Hen97], [Cut97], [Loe97], [HL85], [AHKFL86], [SL76] or [Mar97]. Very recent efforts on axiomatization are exhaustively presented in [KR04].

Most mathematical theory can be formalized in Set Theory. For convenience, we will assume the existence of **atoms**, i.e., objects which are not the empty set but have no elements; however, we will work in the context of Zermelo-Fraenkel Set Theory with the Axiom of Choice.

In order to present our almost axiomatic approach to **NSA**, denote by $\mathcal{X}$ a set containing (models of) all mathematical objects of Classical Analysis which one wishes to study, both as elements and subsets, in case they are sets themselves; namely, $\mathcal{X}$ contains the set of real numbers $\mathbb{R}$, normed linear spaces and all the functionals defined in these spaces. We will consider that the real numbers are atoms in $\mathcal{X}$. If E is a real Banach space we want to study, the elements of E will also be consider as atoms. In addition, we will suppose that we have another set-theoretical structure $\mathcal{Y}$ and an injective map

$$^{\star}(\cdot) : \mathcal{X} \to \mathcal{Y}$$

which satisfies two basic principles: the **Transfer Principle** and the **Polysaturation Principle**.

For each element $a \in \mathcal{X}$, $^{\star}a \in \mathcal{Y}$ will be called the **nonstandard extension** of a.

For convenience, we will assume that the $^{\star}(\cdot)$ mapping is the identity for atoms, that is, if $u \in \mathcal{X}$ is an atom, then $^{\star}u = u$. In particular, $^{\star}r = r$ for all $r \in \mathbb{R}$. Therefore, if X is a set of atoms, we can write $X \subseteq {}^{\star}X$.

In order to formalize the Transfer Principle and the Polysaturation Principle, we will fix a formal first order language $\mathcal{L}$ that contains the logical symbols:

1. *Logical relations:* $=$ (equality relation) and $\in$ (membership relation);
2. *Logical connectives*: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\Longrightarrow$ (implication)

and $\Longleftrightarrow$ (equivalence);

3. *Quantifier symbols*: $\exists$ (existencial quantifier) and $\forall$ (universal quantifier);

4. *Variable symbols*: $x, y, v_1, v_2, \cdots, v_n, \cdots$ (to be used as variables);

5. *Parentheses*: ( ) and [ ] (used as usual in mathematics for bracketing).

Also, the language $\mathcal{L}$ contain enough constants, relation and function symbols to denote any element, relation and function of both structures $\mathcal{X}$ and $\mathcal{Y}$.

With this *alphabet* we can write (*well formed*) formulas in the usual way.

We will always suppose that all the quantifiers that appear in a formula are **bounded**, that is, they have the form

$$\forall x[x \in a \Rightarrow \varphi] \quad \text{or} \quad \exists x[x \in a \wedge \varphi]$$

where x is a variable, a is a set and φ is a formula. These formulas will be abbreviated, respectively, by

$$\forall x \in a \ [\varphi] \quad \text{and} \quad \exists x \in a \ [\varphi].$$

As usual, a variable x is **bounded** if it occurs in the scope of a quantifier ($\forall x$ or $\exists x$); x is **free** otherwise.

A **sentence** is a formula without free variables.

We are now able to present the **Transfer Principle**:

> ***(T):*** *A sentence $\varphi(a_1, \cdots, a_n)$ whose only constants are $a_1, \cdots, a_n$ is true in $\mathcal{X}$ if and only if $\varphi(^\star a_1, \cdots, ^\star a_n)$ is true in $\mathcal{Y}$.*

We will distinguish important subclasses in $\mathcal{Y}$ according to the next definition.

Definition 2.1 *If $a \in \mathcal{Y}$, then*

1. *a is called **standard** if $a = ^\star b$ for some $b \in \mathcal{X}$ (elements of $\mathcal{X}$ are also called standard);*

2. *a is called **internal** if $a \in ^\star b$ for some $b \in \mathcal{X}$;*

3. *a is called **external** if a is not internal.*

We say that a **formula** ϕ in $\mathcal{L}$ is **internal** (respectively **standard**) if all of its constants denote internal (respectively standard) elements of $\mathcal{Y}$.

The following proposition is consequence of the Transfer Principle.

Proposition 2.2 [HL85, page 80] *Let $n \in \mathbb{N}$ and $a, b, a_1, \cdots, a_n$ be sets in $\mathcal{X}$. Then*

1. $^\star\emptyset = \emptyset$;
2. $^\star\{a_1, \cdots, a_n\} = \{^\star a_1, \cdots, ^\star a_n\}$;
3. $a \subseteq b \Leftrightarrow ^\star a \subseteq ^\star b$;
4. $^\star(a \setminus b) = ^\star a \setminus ^\star b$;
5. $^\star(a_1 \times \cdots \times a_n) = ^\star a_1 \times \cdots \times ^\star a_n$;
6. $^\star(\bigcup_{i=1}^n a_i) = (\bigcup_{i=1}^n {}^\star a_i)$ *and* $^\star(\bigcap_{i=1}^n a_i) = (\bigcap_{i=1}^n {}^\star a_i)$;
7. *f is a function from a to b if and only if $^\star f$ is a function from $^\star a$ to $^\star b$.*

Remark 2.3 .

- Condition *2.* of Proposition 2.2 remains true if $a_1, \cdots, a_n$ are atoms in $\mathcal{X}$.
- All standard elements are internal since, for each $a \in \mathcal{X}$, $^\star a \in ^\star\{a\} = \{^\star a\}$.
- For simplicity, we shall often avoid the use of * on standard functions, whenever there is no ambiguity.

Another easy consequence of the Transfer Principle is the following theorem.

Theorem 2.4 [Nev01] *Let A, B be sets in $\mathcal{X}$ and $\varphi(t, a_1, \cdots, a_n)$ is a formula in $\mathcal{L}$ where t is the only free variable and $a_1, \cdots, a_n$ are all the constants that occurs in φ. Then*

$$A = \{t \in B \ : \ \varphi(t, a_1, \cdots, a_n)\} \Leftrightarrow \ ^\star A = \{t \in ^\star B \ : \ \varphi(t, ^\star a_1, \cdots, ^\star a_n)\}.$$

If E is a set, $\mathcal{P}(E)$ will denote the set of subsets of E and $card(E)$ the cardinality of E. We will say that a family $\mathcal{C}$ of sets satisfies the **finite intersection property** if intersections of finite subfamilies of $\mathcal{C}$ are non empty.

The **Polysaturation Principle** reads as follows,

> ***(P):*** *Let E be a set in $\mathcal{X}$ and $\mathcal{C}$ be a collection of internal subsets of $^\star E$. If $\mathcal{C}$ verifies the finite intersection property and $card(\mathcal{C}) < card(\mathcal{X})$, then $\mathcal{C}$ has non empty intersection.*

This is a very strong kind of *compactness property*; actually for some applications, such as the study of Loeb measure theory (Section 2.6), we only need $\aleph_1$**-saturation** (also called **countable saturation**):

> $\aleph_1$***-saturation:*** *Let E be a set in $\mathcal{X}$ and $(A_n)_{n\in\mathbb{N}}$ a sequence of internal subsets of $^\star E$ that satisfies the finite intersection property. Then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.*

In the following theorem we present an alternative formulation of $\aleph_1$-saturation that is very useful in the construction of Loeb measures.

Theorem 2.5 [Lin88, page 13] *Let E be a set in $\mathcal{X}$ and $(A_n)_{n\in\mathbb{N}}$ be a sequence of internal subsets of $^\star E$. Then $\bigcup_{n\in\mathbb{N}} A_n$ is internal if and only if there exists $k \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} A_n = A_1 \cup \cdots \cup A_k$.*

The next two theorems are basic tools for distinction of standard, internal and external sets ([Nev01]).

Theorem 2.6 ***(Internal Definition Principle)*** *Let A be a set in $\mathcal{X}$ and $\phi(x)$ an internal formula in $\mathcal{L}$ where x is the only free variable. Then the set*

$$B := \{x \in {}^\star A : \ \phi(x)\}$$

is internal. Conversely, every internal set can be defined this way.

Theorem 2.7 ***(Standard Definition Principle)*** *Let A be a set in $\mathcal{X}$ and $\phi(x)$ a standard formula in $\mathcal{L}$ where x is the only free variable. Then the set*

$$B := \{x \in {}^{\star}A : \ \phi(x)\}$$

is standard. Conversely, every standard set can be defined this way.

2.3 Hyperreal numbers

Denote by ${}^{\star}\mathbb{R}$ the nonstandard extension of the set of real numbers, $\mathbb{R}$. A simple application of the Transfer Principle shows that ${}^{\star}\mathbb{R}$ is an ordered field under the extension of the operations $+$ and $\cdot$, and the relation $<$. The Polysaturation Principle shows that ${}^{\star}\mathbb{R}$ is a proper extension of $\mathbb{R}$.

The elements of ${}^{\star}\mathbb{R}$ are called **hyperreal numbers** or **nonstandard real numbers** and are classified the following way, where $\mathbb{N}$ denotes the set of natural numbers and $|\cdot|$ is the extension of the absolute value function to ${}^{\star}\mathbb{R}$.

Definition 2.8 *A hyperreal number x is*

1. **infinitesimal** *if $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$ (the set of infinitesimal hyperreal numbers will be denoted ${}^{\star}\mathbb{R}_0$);*

2. **finite** *if $|x| < n$ for some $n \in \mathbb{N}$ (the set of finite hyperreal numbers will be denoted ${}^{\star}\mathbb{R}_{fin}$);*

3. **infinite** *if it is not finite (the set of infinite hyperreal numbers will be denoted ${}^{\star}\mathbb{R}_{\infty}$).*

If $x \in {}^{\star}\mathbb{R}$ is infinitesimal we write $x \approx 0$. If $x, y \in {}^{\star}\mathbb{R}$ and $x - y \approx 0$, we say that x is **infinitely close** to y and write $x \approx y$.

Remark 2.9 The Transfer Principle implies that every nonempty internal subset of ${}^{\star}\mathbb{R}$ that is bounded above (respectively below) has a supremum (respectively infimum); therefore we may conclude that the sets ${}^{\star}\mathbb{R}_0$, ${}^{\star}\mathbb{R}_{fin}$ and ${}^{\star}\mathbb{R}_{\infty}$ are external.

Next we present an important property of the finite hyperreal numbers ([HL85, pages 26-27]).

Theorem 2.10 ***(Standard Part Theorem)*** *If $x \in {}^{\star}\mathbb{R}_{fin}$, there exists a unique $r \in \mathbb{R}$ such that $x \approx r$; r is called the **standard part** of x and is denoted by $st(x)$ or $^{\circ}x$. Moreover, for all $x, y \in {}^{\star}\mathbb{R}_{fin}$, $st(x+y) = st(x) + st(y)$, $st(xy) = st(x)st(y)$ and if $x \leq y$ then $st(x) \leq st(y)$.*

In particular, it follows from the Standard Part Theorem that 0 is the unique infinitesimal real number.

The nonstandard extension of $\mathbb{N}$, ${}^{\star}\mathbb{N}$, will be called the set of **hypernatural numbers** and we will denote by ${}^{\star}\mathbb{N}_{\infty}$ the set of infinite hypernatural numbers.

Definition 2.11 *If H is a set in $\mathcal{Y}$, then H is **hyperfinite** if there exists $\omega \in {}^{\star}\mathbb{N}$ and an internal bijection $f : H \to \{n \in {}^{\star}\mathbb{N} : \ n \leq \omega\}$; ω is called the **internal cardinality** of H.*

Denoting $\mathcal{P}_F(E)$ the set of finite subsets of the set E, it follows that

Proposition 2.12 [HL85, page 89] *H is an hyperfinite subset of ${}^{\star}E$ if and only if $H \in {}^{\star}\mathcal{P}_F(E)$.*

Remark 2.13 Again, the Transfer Principle shows that every hyperfinite set satisfies the same properties of the finite sets as far as they can be formalized in $\mathcal{L}$. For example,

- every hyperfinite set of hyperreal numbers has a minimum and a maximum element;
- every internal subset of a hyperfinite set is also hyperfinite;
- the collection of all internal subsets of a hyperfinite set is hyperfinite;
- internal images of hyperfinite sets are hyperfinite.

Definition 2.14 *If X is a set in $\mathcal{X}$, the **standard copy of** X is the set*

$$^{\sigma}X := \{{}^{\star}x : \ x \in X\}.$$

Therefore, ${}^{\sigma}X$ is the set of all standard elements of ${}^{\star}X$. Notice that if X is a set of atoms, then ${}^{\sigma}X = X$. It may happen that ${}^{\sigma}X \neq {}^{\star}X$, as we can see with the next consequence of the Transfer and Polysaturation Principles ([Nev01]).

Theorem 2.15 ***(Discretization Principle)*** *For any set $X \in \mathcal{X}$, there exists an hyperfinite set $H \in \mathcal{Y}$ such that*

$$ {}^{\sigma}X \subseteq H \subseteq {}^{\star}X. $$

X is infinite if and only if both inclusions are strict.

Other consequences of the Transfer Principle are the Overflow and Underflow Principles described below ([Lin88, page 12]).

Theorem 2.16 ***(Overflow Principle)*** *Let $A \subseteq {}^{\star}\mathbb{R}$ be an internal set. If A contains arbitrarily large finite numbers, then A contains an infinite number. If A contains arbitrarily large positive infinitesimal numbers, then A contains a positive finite number which is not infinitesimal.*

Theorem 2.17 ***(Underflow Principle)*** *Let $A \subseteq {}^{\star}\mathbb{R}$ be an internal set. If A contains arbitrarily small positive infinite numbers, then A contains a positive finite number. If A contains arbitrarily small positive non infinitesimal numbers, then A contains a positive infinitesimal number.*

The following is an important consequence of the Polysaturation Principle ([Nev01]).

Theorem 2.18 ***(Comprehension Principle)*** *Suppose that X and Y are sets in $\mathcal{X}$, $A \subseteq {}^{\star}X$, $B \subseteq {}^{\star}Y$, $card(A) < card(\mathcal{X})$ and B is internal. For each $f : A \to B$, there exists an internal function $g : {}^{\star}X \to B$ such that $g|_A = f$.*

2.4 Topology

Throughout this section, $(E, \|\cdot\|)$ will denote a real normed space.

Definition 2.19 *If $x \in {}^\star E$ then*

1. *x is **finite** if $\| x \| \in {}^\star\mathbb{R}_{fin}$; the set of finite elements of ${}^\star E$ shall be denoted $fin({}^\star E)$;*

2. *x is **infinitesimal** if $\| x \| \approx 0$ in ${}^\star\mathbb{R}$; denote by $inf({}^\star E)$ the set of infinitesimal elements of ${}^\star E$ and write $x \approx 0$ for $x \in inf({}^\star E)$;*

3. *x is **near-standard** if there exists $a \in E$, called the **standard part** of x, such that $x - a \approx 0$, and we write $x \approx a$; the set of near-standard elements will be denoted $ns({}^\star E)$;*

4. *x is **pre-near-standard** if $\forall \epsilon \in \mathbb{R}^+ \; \exists a \in E \;\; \| x - a \| < \epsilon$; the set of pre-near-standard elements shall be denoted $pns({}^\star E)$.*

As usual, if $x, y \in {}^\star E$ are such that $x - y \approx 0$, we say that x is **infinitely close** to y and write $x \approx y$; otherwise, we write $x \not\approx y$.

Definition 2.20 *For each $a \in E$, the set*

$$mon(a) := \{x \in {}^\star E : \; x \approx a\}$$

*is called the **monad** of a.*

Theorem 2.21 [HL85, page 114] *If $a, b \in E$ are such that $a \neq b$, then $mon(a) \cap mon(b) = \emptyset$.*

From the last theorem it makes sense to define the following application

$$\begin{array}{rcl} st : ns({}^\star E) & \rightarrow & E \\ x & \mapsto & st(x) \end{array}$$

called the **standard part function**, where $st(x)$ is the unique $a \in E$ such that $a \approx x$. The notation ${}^\circ x$ for $st(x)$ is often used.

If $Y \subseteq {}^\star E$, define

$$st(Y) := \{st(x) : \; x \in Y \cap \; ns({}^\star E)\}.$$

In the sequel we will describe nonstandard characterizations of some topological concepts.

Theorem 2.22 [Lin88, pages 52-53] *Let $(E, \|\cdot\|)$ be a real normed space and $A \subseteq E$.*

1. *A is open if and only if for all $a \in A$, $mon(a) \subseteq {}^\star A$;*
2. *A is closed if and only if $st({}^\star A) = A$;*
3. *$A \subseteq E$ is compact if and only if ${}^\star A \subseteq ns({}^\star E)$ and $st({}^\star A) = A$.*

Characterizations of some properties of sequences in E are as follows.

Proposition 2.23 [Dav77, pages 91-92] *Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E. Then*

1. *$(x_n)_{n\in\mathbb{N}}$ is bounded if and only if $\forall n \in {}^\star\mathbb{N}_\infty\ x_n \in fin({}^\star E)$;*
2. *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\forall n, m \in {}^\star\mathbb{N}_\infty\ x_n \approx x_m$;*
3. *$(x_n)_{n\in\mathbb{N}}$ converges to $a \in E$ if and only if $\forall n \in {}^\star\mathbb{N}_\infty\ x_n \approx a$;*
4. *$(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence if and only if $\exists m \in {}^\star\mathbb{N}_\infty\ x_m \in ns({}^\star E)$.*

From Definition 2.19 it follows that $ns({}^\star E) \subseteq fin({}^\star E) \cap pns({}^\star E)$. But, in general, $ns({}^\star E) \neq fin({}^\star E)$ and $ns({}^\star E) \neq pns({}^\star E)$ as we will see in the following result ([Dav77, page 90], [HL85, page 127]) and Examples 2.25 and 2.26.

Theorem 2.24 *Let $(E, \|\cdot\|)$ be a real normed space.*

1. *E is finite dimensional if and only if $fin({}^\star E) = ns({}^\star E)$;*
2. *E is complete if and only if $ns({}^\star E) = pns({}^\star E)$.*

Example 2.25 Consider the real normed space

$$X = l^1(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ :\ \sum_{n\in\mathbb{N}} \mid x_n \mid < \infty\}$$

where, for each $x = (x_n)_{n\in\mathbb{N}} \in X$,

$$\| x \| = \sum_{n\in\mathbb{N}} | x_n | .$$

Take $\omega \in {}^{\star}\mathbb{N}_{\infty}$ and define

$$\begin{aligned} g : {}^{\star}\mathbb{N} &\rightarrow {}^{\star}\mathbb{R} \\ n &\mapsto g_n = \begin{cases} \frac{1}{\omega} & \text{if} \quad 1 \le n \le \omega \\ 0 & \text{if} \quad n > \omega \end{cases} . \end{aligned}$$

Since

$$\| g \| = \sum_{n\in{}^{\star}\mathbb{N}} | g_n | = \sum_{n=1}^{\omega} \frac{1}{\omega} = 1$$

it follows that $g \in fin({}^{\star}X)$. Next we will prove that $g \notin ns({}^{\star}X)$. Suppose that there exists $a = (a_n)_{n\in\mathbb{N}} \in X$ such that $g \approx a$. Since

$$\begin{aligned} \| g - a \| \approx 0 &\Leftrightarrow \textstyle\sum_{n\in{}^{\star}\mathbb{N}} | g_n - a_n | \approx 0 \\ &\Rightarrow \forall n \in {}^{\star}\mathbb{N} \; g_n \approx a_n \\ &\Rightarrow \forall n \in \mathbb{N} \; a_n = 0 \\ &\Leftrightarrow \| a \| = 0 \end{aligned}$$

and

$$\| g - a \| \ge \| g \| - \| a \| = 1$$

we obtain a contradiction with $g \approx a$; hence $g \notin ns({}^{\star}X)$.

Example 2.26 .

1. Let $\mathbb{Q}$ denote the set of rational numbers, $x = (x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ that converges to some $a \in \mathbb{R} \setminus \mathbb{Q}$ and $\omega \in {}^{\star}\mathbb{N}_{\infty}$. Since $x_\omega \approx a$ (Proposition 2.23) and $a \notin \mathbb{Q}$ then $x_\omega \notin ns({}^{\star}\mathbb{Q})$ (Theorem 2.21). On the other hand, $x_\omega \in pns({}^{\star}\mathbb{Q})$, because in any open interval centered in x_ω with radius $\epsilon \in \mathbb{R}^+$ there exists a rational number.

2. Let $X = (C([-2, 2]), \| \cdot \|)$ be the (incomplete) normed space of all continuous real valued functions defined in $[-2, 2]$ with the norm $\| \cdot \|$ defined by

$$\| f \| = \int_{-2}^{2} | f(t) | \, dt \qquad (f \in C([-2, 2]).$$

Consider the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in X defined by

$$f_n(t) = \begin{cases} 0 & \text{if} \quad -2 \leq t < -\frac{1}{n} \\ \frac{n}{2}t + \frac{1}{2} & \text{if} \quad -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} < t \leq 2 \end{cases}$$

and $g : [-2, 2] \to \mathbb{R}$ be such that

$$g(t) = \begin{cases} 0 & \text{if} \quad -2 \leq t < 0 \\ \frac{1}{2} & \text{if} \quad t = 0 \\ 1 & \text{if} \quad 0 < t \leq 2 \end{cases} .$$

Observe that, for any $\omega \in {}^\star\mathbb{N}_\infty$,

$$\| f_\omega - g \| = \int_{-2}^{2} | f_\omega(t) - g(t) | \, dt = 2\int_0^{\frac{1}{\omega}} \left(1 - \left(\frac{\omega}{2}t + \frac{1}{2}\right)\right) dt = \int_0^{\frac{1}{\omega}} (1 - \omega t) dt = \frac{1}{2\omega} \approx 0.$$

Since $g \notin C([-2, 2])$, then $f_\omega \notin ns({}^\star X)$ (Theorem 2.21).

Next we will show that $f_\omega \in pns({}^\star X)$. Notice that, for each $n \in \mathbb{N}$,

$$\| f_\omega - f_n \| = 2\int_0^{\frac{1}{\omega}} \left(\frac{\omega}{2}t - \frac{n}{2}t\right) dt + 2\int_{\frac{1}{\omega}}^{\frac{1}{n}} \left(1 - \frac{n}{2}t - \frac{1}{2}\right) dt = \frac{1}{2n} - \frac{1}{2\omega} < \frac{1}{2n}.$$

Take $\epsilon \in \mathbb{R}^+$; choosing $m \in \mathbb{N}$ such that $\frac{1}{2m} < \epsilon$ we have that $\| f_\omega - f_m \| < \epsilon$, and this shows that $f_\omega \in pns({}^\star X)$. Hence, $ns({}^\star X) \neq pns({}^\star X)$.

In the following, we will suppose that $(F, | \cdot |)$ is another real normed space.

Definition 2.27 *Let $g : {}^\star E \to {}^\star F$ be a function. Then g is said to be* ***S-continuous*** *on a (possibly external) subset A of its domain if*

$$\forall x, y \in A \ [x \approx y \Rightarrow g(x) \approx g(y)].$$

Some relations between this notion and the usual continuity are the following results.

Theorem 2.28 [HL85, pages 115 and 125] *A function $f : E \to F$ is*

1. *continuous on $c \in E$ if and only if it is S-continuous in $mon(c)$;*

2. *continuous if and only if f is S-continuous in $ns(^\star E)$;*

3. *uniformly continuous if and only if f is S-continuous in $^\star E$.*

In the sequel we present some results that will be very useful in Chapter 3, when we will study some Existence Theorems for **ODE's**.

Theorem 2.29 [Cut97, page 72] *If $[a,b] \subseteq \mathbb{R}$, $g : {}^\star[a,b] \to {}^\star\mathbb{R}$ is internal and S-continuous and there exists $z \in {}^\star[a,b]$ such that $g(z)$ is finite, then*

1. *$g(x)$ is finite for all $x \in {}^\star[a,b]$;*

2. *the standard function*
$$\begin{array}{rcl} {}^\circ g : [a,b] & \to & \mathbb{R} \\ t & \mapsto & {}^\circ g(t) := st(g(t)) \end{array}$$
is continuous.

Remark 2.30 Theorem 2.29 remains true if we substitute ${}^\star[a,b]$ by a hyperfinite set $\mathbb{X}$ such that $st(\mathbb{X}) = [a,b]$; in this case, ${}^\circ g : [a,b] \to \mathbb{R}$ is such that, for each $t \in [a,b]$,

$${}^\circ g(t) := st(g(\tau))$$

for some $\tau \in \mathbb{X}$ satisfying $\tau \approx t$.

Definition 2.31 *Let $Y \subseteq {}^\star E$ and $g : Y \to {}^\star\mathbb{R}$. Then g is* ***S-bounded*** *if there exists $L \in \mathbb{R}$ such that for all $y \in Y$,*

$$\mid g(y) \mid \leq L.$$

Definition 2.32 *Let $Y \subseteq {}^\star E$ and $g : Y \to {}^\star\mathbb{R}$. Then g is* ***S-Lipschitzian*** *if there exists $M \in \mathbb{R}$ such that for all $x, y \in Y$,*

$$\mid g(x) - g(y) \mid \leq M \parallel x - y \parallel .$$

Definition 2.33 *Let $Y \subseteq {}^\star\mathbb{R}$ and $g : Y \to {}^\star\mathbb{R}$. Then g is* ***S-absolutely continuous*** *if*

$$\sum_{i=1}^{N} |g(b_i) - g(a_i)| \approx 0$$

for every hyperfinite collection

$$\{[a_1, b_1[, [a_2, b_2[, \cdots, [a_N, b_N[\} \qquad (N \in {}^\star\mathbb{N})$$

(where $[a, b[$ denotes the set $\{t \in {}^\star\mathbb{R} : a \leq t < b\} \cap Y$ and $a, b \in Y$) of non overlapping subintervals of Y such that $\sum_{i=1}^{N}(b_i - a_i) \approx 0$.

The following is easy to prove.

Proposition 2.34 *Let $Y \subseteq {}^\star\mathbb{R}$ and $g : Y \to {}^\star\mathbb{R}$.*

1. *If g is S-absolutely continuous, then g is S-continuous;*
2. *If g is S-Lipschitzian, then g is S-absolutely continuous.*

Next we present an important result that relates the (nonstandard) notion of S-absolutely continuity and the (classical) notion of absolutely continuity. We recall that a function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** on $[a, b]$ if, for any $\epsilon \in \mathbb{R}^+$, there is a $\delta \in \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} \mid f(b_i) - f(a_i) \mid < \epsilon$$

for any disjoint intervals $[a_1, b_1[, \cdots, [a_n, b_n[$ in $[a, b]$ whose lengths satisfy

$$\sum_{i=1}^{n}(b_i - a_i) < \delta.$$

Theorem 2.35 [Tuc93, pages 35-36] *Let $[a, b] \subseteq \mathbb{R}$ and $\mathbb{X}$ a hyperfinite set such that $st(\mathbb{X}) = [a, b]$. If $g : \mathbb{X} \to {}^\star\mathbb{R}$ is internal, S-bounded and S-absolutely continuous function, then the function ${}^\circ g : [a, b] \to \mathbb{R}$ defined by*

$${}^\circ g(t) := st(g(\tau))$$

where $\tau \in \mathbb{X}$ is such that $\tau \approx t$, is a standard absolutely continuous function.

2.5 Differentiation

Suppose $(E, \| \cdot \|)$ and $(F, | \cdot |)$ are real Banach spaces and let $Lin(E,F)$ denote the space of all continuous linear maps from E to F. Let U be an open subset of E and $f : U \to F$ a map. We say that $f : U \to F$ is a C^1 function and write $f \in C^1(U,F)$, if the Fréchet derivative $f' : U \to Lin(E,F)$ exists at every point $a \in U$ and the mapping f' is continuous.

The next result follows easily from the nonstandard characterization of limits.

Theorem 2.36 *The map $f : U \to F$ is Fréchet differentiable if and only if there exists an application $f' : U \to Lin(E,F)$ such that*

$$\forall a \in U \ \forall \epsilon \in inf({}^\star E) \ \exists \eta \in inf({}^\star F) \ f(a+\epsilon) = f(a) + f'(a)(\epsilon) + \parallel \epsilon \parallel \eta.$$

Next we present nonstandard characterizations of C^1 functions. Denote

$$ns({}^\star U) := \{x \in {}^\star E : \ x \in ns({}^\star E) \ \wedge \ st(x) \in U\}.$$

Theorem 2.37 [SL76, page 97] *Suppose $f : U \to F$ is a map. The following conditions are equivalent:*

1. *$f \in C^1(U,F)$;*
2. *there exists a map $f' : U \to Lin(E,F)$ such that*
$$\forall a \in ns({}^\star U) \ \forall \epsilon \in inf({}^\star E) \ \exists \eta \in inf({}^\star F) \quad f(a+\epsilon) = f(a) + f'(a)(\epsilon) + \parallel \epsilon \parallel \eta;$$
3. *there exists a map $f' : U \to Lin(E,F)$ such that*
$$\forall a \in U \ \forall x, y \in {}^\star E \ [\ x \approx y \approx a \ \Rightarrow$$
$$\exists \beta \in inf({}^\star F) \ f(x) - f(y) = f'(a)(x-y) + \parallel x - y \parallel \beta \ ].$$

We notice that the only difference between the nonstandard characterization of a Fréchet differentiable map (Theorem 2.36) and the nonstandard characterization of a C^1 map given in condition *2.* of Theorem 2.37, is "only" the set to which a belongs: in the first case, $a \in U$ and in the second case, $a \in ns({}^\star U)$.

We proceed with other nonstandard characterization of a C^1 map (in the sense of Fréchet) using the weaker notion of Gâteaux-Levy derivative.

Theorem 2.38 [Str78, pages 367-368] *Suppose $f : U \to F$ is a map. Then $f \in C^1(U,F)$ if and only if there exists a map $Df_{(\cdot)} : U \to Lin(E,F)$ such that*

$$\forall a \in ns(^\star U)\ \forall x \in fin(^\star E)\ \forall \epsilon \in {}^\star\mathbb{R}_0\ \exists \beta \in inf(^\star F)\ \ f(a+\epsilon x) = f(a) + \epsilon Df_a(x) + \epsilon\beta.$$

Denote by E' the dual space of E with uniform norm $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ the duality pairing between E' and E. For later use, we present the following definition.

Definition 2.39 *Let $f : U \to \mathbb{R}$ be Fréchet differentiable and $a \in {}^\star U$. Then a is an* ***almost critical point*** *of f if $f'(a) \approx 0$.*

Therefore, a is an almost critical point of $f : U \to \mathbb{R}$ if $\| f'(a) \| \approx 0$, that is,

$$\forall v \in fin(^\star E)\ \ \langle f'(a), v \rangle \approx 0.$$

2.6 Loeb integration theory

Loeb measures were discovered by Peter Loeb in 1975 ([Loe75]) and they are very important in many applications of **NSA**. These measures have appeared in Control Theory, Mathematical Physics, Mathematical Finance, Functional Analysis and several other fields. In Chapter 3 we will apply (finite) Loeb measures to ordinary differential equations, namely, we will present a nonstandard proof of Carathéodory's Existence Theorem.

For convenience, we will describe only the construction of finite Loeb measures. These measures are obtained from an internal measure in the way described below. For a more complete treatment of nonstandard integration theory see [Cut83], [Cut95], [Cut00], [Lin88], [SB86], [Ros97], [And82], [AHKFL86] or [Mar97], for example.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is a **finite internal measure space**, that is,

1. Ω is an internal non empty set;

2. $\mathcal{A}$ is an internal algebra on Ω (that is, $\mathcal{A}$ is internal, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, $\Omega \in \mathcal{A}$ and $\mathcal{A}$ is closed for complements and finite unions);

3. $\mu : \mathcal{A} \rightarrow {}^{\star}\mathbb{R}$ is a finite internal finitely additive measure (that is, μ is an internal function such that $\mu(\Omega)$ is finite, $\mu(\emptyset) = 0$, $\mu(A) \geq 0$ for all $A \in \mathcal{A}$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint $A, B \in \mathcal{A}$).

In general, this is not a measure space because $\mathcal{A}$ is not a σ-algebra, except in the trivial case where $\mathcal{A}$ is finite. This is a consequence of $\aleph_1$-saturation: if $\mathcal{A}$ is infinite, there exists a countable collection of pairwise disjoint non empty sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$; Theorem 2.5 shows that $\bigcup_{n\in\mathbb{N}} A_n$ is not internal and, therefore, $\bigcup_{n\in\mathbb{N}} A_n \notin \mathcal{A}$.

From $\mu : \mathcal{A} \rightarrow {}^{\star}\mathbb{R}$ we can define the (external) mapping

$$\begin{array}{rcl} {}^{\circ}\mu : \mathcal{A} & \rightarrow & \mathbb{R} \\ A & \mapsto & {}^{\circ}\mu(A) := {}^{\circ}(\mu(A)). \end{array}$$

The Loeb measure generated by μ will be denoted by μ_L and is a measure defined in a family of subsets of Ω that contains the internal algebra $\mathcal{A}$ and that coincide with ${}^{\circ}\mu$ on $\mathcal{A}$.

Definition 2.40 *Let $B \subseteq \Omega$ (B not necessarily internal). We say that*

1. *B is a **Loeb null set** if for each real $\epsilon > 0$ there exists an internal set $A \in \mathcal{A}$ such that $B \subseteq A$ and $\mu(A) < \epsilon$;*

2. *B is **Loeb measurable** if there exists a set $A \in \mathcal{A}$ such that $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is Loeb null. Denote the collection of all Loeb measurable sets by $L(\mathcal{A})$;*

3. *For $B \in L(\mathcal{A})$ define*

$$\mu_L(B) := {}^{\circ}\mu(A)$$

*for all $A \in \mathcal{A}$ such that $A \triangle B$ is Loeb null; $\mu_L(B)$ is called the **Loeb measure** of B.*

Remark 2.41 Observe that

- A subset of a Loeb null set is a Loeb null set;
- $\mu_L : L(\mathcal{A}) \to \mathbb{R}_0^+$;
- $\forall A \in \mathcal{A} \;\; \mu_L(A) = {}^\circ\mu(A)$.

The next proposition clarifies the term *Loeb null set*.

Proposition 2.42 [Cut95, page 155] *For any $B \subseteq \Omega$, B is a Loeb null set if and only if $B \in L(\mathcal{A})$ and $\mu_L(B) = 0$.*

The next proposition says that a set $Y \subseteq \Omega$ is Loeb measurable if it is *almost internal* in the sense described below.

Proposition 2.43 [Mar97, pages 47-48] *$Y \in L(\mathcal{A})$ if and only if there exists $C \in \mathcal{A}$ and a Loeb null set N such that $Y = C \triangle N$.*

In [Cut95], [Cut00], [SB86], [AHKFL86] and [Mar97] the reader can find alternative characterizations of Loeb measurable sets.

Now we present the central theorem in Loeb measure theory.

Theorem 2.44 [Cut95, pages 155-156] *$L(\mathcal{A})$ is a σ-algebra, called **Loeb σ-algebra**, and μ_L is a complete σ-additive measure on $L(\mathcal{A})$.*

$(\Omega, L(\mathcal{A}), \mu_L)$ is a measure space, called the **Loeb measure space** generated by $(\Omega, \mathcal{A}, \mu)$.

Remark 2.45 Note that

- μ_L acts on sets which may not be standard;
- A countable union of Loeb null sets is a Loeb null set.

Remark 2.46 If μ is not finite, it is also possible to define the (unbounded) Loeb measure space $(\Omega, L(\mathcal{A}), \mu_L)$ generated by $(\Omega, \mathcal{A}, \mu)$; see [Lin88] or [Mar97].

In the following we present important examples of Loeb spaces.

Example 2.47 Let $(X, \mathcal{L}, \lambda)$ be a standard measure space and take $\Omega = {}^\star X$, $\mathcal{A} = {}^\star\mathcal{L}$ and $\mu = {}^\star\lambda$. Then $({}^\star X, L({}^\star\mathcal{L}), {}^\star\lambda_L)$ is the Loeb space generated by $({}^\star X, {}^\star\mathcal{L}, {}^\star\lambda)$.

Example 2.48 Fix $N \in {}^\star\mathbb{N}_\infty$, define $\triangle = \frac{1}{N}$ and make

$$\mathbb{T} := \{k\triangle : \ k = 0, 1, 2, \cdots, N-1\} = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1 - \frac{1}{N}\}. \tag{2.1}$$

$\mathbb{T}$ is usually called **hyperfinite time line** with increment $\triangle$. Denoting the set of all internal subsets of $\mathbb{T}$ by $\mathcal{A}$ and defining $\nu : \mathcal{A} \to {}^\star[0,1]$ by

$$\nu(A) = \frac{card(A)}{card(\mathbb{T})} = \frac{card(A)}{N} \tag{2.2}$$

we obtain an internal measure space $(\mathbb{T}, \mathcal{A}, \nu)$ called **internal counting measure space on** $\mathbb{T}$. The Loeb space $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ generated by $(\mathbb{T}, \mathcal{A}, \nu)$ is called the **Loeb counting measure space on** $\mathbb{T}$.

Next we will see how the Loeb counting measure space on the hyperfinite time line can be used to represent the Lebesgue measure space $([0,1], \mathcal{L}, \lambda)$.

Theorem 2.49 [Cut00, page 17] *Let $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ be the Loeb counting measure space. A set $A \subseteq [0,1]$ is Lebesgue measurable if and only if the set $st_{\mathbb{T}}^{-1}(A)$ defined by*

$$st_{\mathbb{T}}^{-1}(A) := \{t \in \mathbb{T} : \ {}^\circ t \in A\}$$

is Loeb measurable; if this is the case,

$$\lambda(A) = \nu_L(st_{\mathbb{T}}^{-1}(A)).$$

Remark 2.50 Theorem 2.49 is a particular case of a general *hyperfinite representation theorem* due to Anderson ([And82]) that shows that any Radon measure space on a Hausdorff space can be represented by a hyperfinite Loeb counting measure.

We deal now with measurable functions.

Definition 2.51 *A function $f : \Omega \to \mathbb{R}$ is **Loeb measurable** if f is μ_L-measurable in the conventional sense, that is, for every open set $B \subseteq \mathbb{R}$, $f^{-1}(B) \in L(\mathcal{A})$.*

Definition 2.52 *An internal function $F : \Omega \to {}^\star\mathbb{R}$ is ***measurable** if $F^{-1}(A) \in \mathcal{A}$, for any *open set $A \subseteq {}^\star\mathbb{R}$ (that is, $A \in {}^\star\mathcal{O}$ where $\mathcal{O}$ denotes the euclidian topology of $\mathbb{R}$).*

Connections between these notions are given in the following theorem.

Theorem 2.53 [Cut95, page 166] *If $F : \Omega \to {}^\star\mathbb{R}$ is internal and *measurable, then the function*

$$\begin{array}{rcl} {}^\circ F : \Omega & \to & \mathbb{R} \\ w & \mapsto & {}^\circ F(w) := {}^\circ(F(w)). \end{array}$$

is Loeb measurable.

Now we present important notions in nonstandard integration theory. As usual in measure theory, we will write **a.a.** to mean "almost all".

Definition 2.54 .

1. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and $f : \Omega \to \mathbb{R}$. An internal *measurable function $F : \Omega \to {}^\star\mathbb{R}$ is called a (one legged) **lifting** of f if*

$$ {}^\circ F(w) = f(w) \quad \mu_L - a.a.\ w \in \Omega.$$

$$\begin{tikzcd} \Omega \arrow[r, "F"] \arrow[dr, "f"'] & {}^\star\mathbb{R} \arrow[d, "\circ"] \\ & \mathbb{R} \end{tikzcd}$$

2. *Let $(X, \mathcal{L}, \lambda)$ be a standard measure space and $f : X \to \mathbb{R}$. An internal *measurable function $F : {}^\star X \to {}^\star\mathbb{R}$ is a (two legged)* ***lifting*** *of f if*

$$^\circ F(x) = f(\,^\circ x) \quad {}^\star\lambda_L - a.a.\ x \in {}^\star X.$$

$$\begin{tikzcd} {}^\star X \arrow[r, "F"] \arrow[d, "\circ"'] & {}^\star\mathbb{R} \arrow[d, "\circ"] \\ X \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

The basic result about measurability is the following.

Theorem 2.55 [Cut95, page 166] *Let $(\Omega, \mathcal{A}, \mu)$ be a finite internal measure space and $f : \Omega \to \mathbb{R}$. Then f is Loeb measurable if and only if f has a lifting $F : \Omega \to {}^\star\mathbb{R}$. If f is bounded above (or below) then F may be chosen with the same bound.*

Remark 2.56 If we remove the assumption that $\mu(\Omega)$ is finite, the last theorem is false (see [Lin88, page 37]): there are Loeb measurable functions with no lifting.

Next we will present a result that may be considered the main lemma for our nonstandard proof of Carathéodory's Existence Theorem (Theorem 3.3).

Theorem 2.57 ***(Anderson's Theorem)*** *Let $(X, \mathcal{L}, \lambda)$ be a Lebesgue measure space, (Y, Γ) a Hausdorff space with a countable base of open sets and $f : X \to Y$ a Lebesgue measurable function. Then ${}^\star f$ is a lifting of f in the following sense:*

1. *$({}^\star f)^{-1}(A) \in {}^\star\mathcal{L}$ for any $A \in {}^\star\Gamma$*
2. *${}^\star f(x) \approx f(^\circ x) \quad {}^\star\lambda_L - a.a.\ x \in {}^\star X.$*

Remark 2.58 Anderson ([And82, pages 672-673]) proves this result in the case where $(X, \mathcal{L}, \lambda)$ is a complete Radon space.

Loeb measures are classical measures over σ-algebras (with possibly nonstandard elements), thus Loeb integration theory is simply the classical theory of integration with respect to Loeb measure: in particular, a Loeb measurable function $f:\Omega \to \mathbb{R}$ is **Loeb integrable** if f is integrable in the classical sense with respect to the Loeb measure μ_L, in which case the **Loeb integral** $\int_\Omega f d\mu_L$ is a real number.

The ***integral** or **internal integral** of a *measurable function $F:\Omega \to {}^\star\mathbb{R}$ is obtained applying the Transfer Principle to the classical definition of integral; usually we will write $\int F d\mu$ instead ${}^\star\int F d\mu$.

Although Theorem 2.53 says that if $F:\Omega \to {}^\star\mathbb{R}$ is internal and *measurable, then ${}^\circ F$ is Loeb measurable and for all $x \in \Omega$

$$F(x) \approx {}^\circ F(x),$$

the equation

$$^\circ\left(\int_\Omega F d\mu\right) = \int_\Omega {}^\circ F d\mu_L \tag{2.3}$$

is, in general, false.

Example 2.59 Let $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ be the Loeb counting measure space. Define the internal *measurable function

$$F(\tau) = \begin{cases} N & \text{if} \quad \tau = 0 \\ 0 & \text{if} \quad \tau \in \mathbb{T}\setminus\{0\} \end{cases}$$

where $N \in {}^\star\mathbb{N}_\infty$ is the same used in the construction of $\mathbb{T}$. Then $\int_\mathbb{T} {}^\circ F d\nu_L = 0$ (since ${}^\circ F(\tau) = 0$ for ν_L- a.a. $\tau \in \mathbb{T}$) and

$$\int_\mathbb{T} F d\nu = \sum_{\tau\in\mathbb{T}} F(\tau)\nu(\{\tau\}) = \sum_{\tau\in\mathbb{T}} F(\tau)\frac{1}{N} = 1.$$

To obtain equality of ${}^\circ\left(\int_\Omega F d\mu\right)$ and $\int_\Omega {}^\circ F d\mu_L$ we have to restrict the class of *integrable functions.

Definition 2.60 *An internal *measurable function $F:\Omega \to {}^\star\mathbb{R}$ is **S-integrable** if*

1. *$\int_\Omega \mid F \mid d\mu$ is finite;*

2. *for all $A \in \mathcal{A}$ such that $\mu(A) \approx 0$, then $\int_A \mid F \mid d\mu \approx 0$.*

Remark 2.61 Concerning the last definition,

- Condition *1.* is necessary to guarantee that all S-integrable function are *integrable;
- Condition *2.* is needed for equality (2.3), because if $\mu(A) \approx 0$, then $\mu_L(A) = 0$ and therefore, $\int_A {}^\circ \mid F \mid d\mu_L = 0$; so $\int_A \mid F \mid d\mu$ must be infinitesimal;
- If μ is not finite we must add an extra condition to Definition 2.60:

  3. if $A \in \mathcal{A}$ and $F \approx 0$ on A, then $\int_A \mid F \mid d\mu \approx 0$.

  Observe that if μ is finite, condition *3.* is always satisfied, since $F \approx 0$ on A means that

  $$\forall \epsilon \in \mathbb{R}^+ \ \forall x \in A \ \mid F(x) \mid < \epsilon$$

  and then, for all $\epsilon \in \mathbb{R}^+$,

  $$\int_A \mid F \mid d\mu \leq \epsilon\mu(A);$$

  since $\mu(A)$ is finite, it follows that $\int_A \mid F \mid d\mu \approx 0$.

The following is easy to prove.

Proposition 2.62 *Let $(\Omega, \mathcal{A}, \mu)$ be a finite internal measure space and $F : \Omega \rightarrow {}^\star\mathbb{R}$ internal *mensurable. If F is S-bounded, then F is S-integrable.*

The following theorem shows that if $F : \Omega \rightarrow {}^\star\mathbb{R}$ is S-integrable, then equality (2.3) holds.

Theorem 2.63 [Cut95, pages 169-170] *Let $(\Omega, \mathcal{A}, \mu)$ an internal measure space and $F : \Omega \rightarrow {}^\star\mathbb{R}$ an internal *mensurable function. Then the following conditions are equivalent:*

1. *F is S-integrable;*
2. *$^\circ F$ is Loeb integrable and*

$$^\circ\left(\int_A F d\mu\right) = \int_A {}^\circ F d\mu_L, \quad \forall A \in \mathcal{A}.$$

The next result relates the Loeb integral of $f : \Omega \to \mathbb{R}$ to the *integral of a lifting of f.

Theorem 2.64 [Cut95, pages 170-171] *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and $f : \Omega \to \mathbb{R}$ a Loeb measurable function. Then the following conditions are equivalent:*

1. *f is Loeb integrable;*
2. *f has an internal S-integrable lifting $F : \Omega \to {}^\star\mathbb{R}$ and*

$$^\circ\left(\int_A F d\mu\right) = \int_A f d\mu_L, \quad \forall A \in \mathcal{A}.$$

The next theorem characterizes nonstandard extensions of Lebesgue integrable functions.

Theorem 2.65 [And82, pages 679-680] *Let $(Z, \mathcal{L}, \lambda)$ be a Lebesgue measure space and suppose that $f : Z \to \mathbb{R}$ is Lebesgue integrable. Then $^\star f :\ ^\star Z \to\ ^\star\mathbb{R}$ is S-integrable.*

To end this section we will present the following characterization of the Lebesgue integral on $[0,1]$. For $f : [0,1] \to \mathbb{R}$ define

$$\begin{aligned} \widehat{f} : \mathbb{T} &\to \mathbb{R} \\ \tau &\mapsto \widehat{f}(\tau) = f(^\circ\tau). \end{aligned}$$

Theorem 2.66 [Mar97, pages 66-68] *Let $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ be the Loeb counting measure space and $([0,1], \mathcal{L}, \lambda)$ the Lebesgue measure space on $[0,1]$. The following conditions are equivalent:*

1. *$f : [0,1] \to \mathbb{R}$ is Lebesgue integrable;*
2. *$\widehat{f} : \mathbb{T} \to \mathbb{R}$ is Loeb integrable;*
3. *there exists an internal S-integrable function $F : \mathbb{T} \to {}^\star\mathbb{R}$ that is a lifting of f.*

In this case

$$\int_{[0,1]} f(t) d\lambda(t) = \int_{\mathbb{T}} \widehat{f}(\tau) d\nu_L(\tau) =\ ^\circ\left(\int_{\mathbb{T}} F d\nu\right) =\ ^\circ\left(\sum_{\tau\in\mathbb{T}} F(\tau)\frac{1}{N}\right).$$

Remark 2.67 Note that the last theorem defines the Lebesgue integral on $[0,1]$ as the standard part of some hyperfinite sum. This is also true for the Lebesgue integral on $\mathbb{R}$ (see [SB86] or [Mar97] for details).

Example 2.68 It may happen that G and F are two liftings of the same Lebesgue integrable function $f:[0,1]\to\mathbb{R}$ but G is S-integrable (then, $\int_{[0,1]} f(t)d\lambda(t) = {}^\circ\left(\int_{\mathbb{T}} Gd\nu\right)$) and F is not S-integrable. Take $F:\mathbb{T}\to{}^\star\mathbb{R}$ defined in Example 2.59 and let $G:\mathbb{T}\to{}^\star\mathbb{R}$ be such that

$$G(\tau) = \begin{cases} 1 & \text{if} \quad \tau = 0 \\ 0 & \text{if} \quad \tau \in \mathbb{T}\setminus\{0\} \end{cases}.$$

It is clear that F and G are liftings of the null function defined on $[0,1]$. G is S-integrable because G is S-bounded (Proposition 2.62) but F is not S-integrable because (Theorem 2.63)

$$1 = {}^\circ\left(\int_{\mathbb{T}} Fd\nu\right) \neq \int_{\mathbb{T}} {}^\circ F d\nu_L = 0.$$

2.7 Nonstandard discrete derivative

Let $\mathbb{T}$ be the hyperfinite time line with respect to the increment $\triangle = \frac{1}{N}$ and $N \in {}^\star\mathbb{N}_\infty$ (see (2.1)). The **nonstandard discrete derivative** or **hyperfinite difference quotient** ([Tuc93, page 34]) of an internal function $X:\mathbb{T}\to{}^\star\mathbb{R}$ is the function $X':\mathbb{T}\setminus\{1-\triangle\}\to{}^\star\mathbb{R}$ defined by

$$X'(t) := \frac{X(t+\triangle)-X(t)}{\triangle}.$$

We finish this chapter by presenting the following result that will be used in the next chapter.

Theorem 2.69 [Tuc93, pages 34-35] *Let $(\mathbb{T},\mathcal{A},\nu)$ be the internal counting measure space and suppose $X:\mathbb{T}\to{}^\star\mathbb{R}$ is an internal function. The following conditions are equivalent:*

1. *X is S-absolutely continuous;*

2. *$\int_A \mid X' \mid d\nu = \sum_{\tau\in A} \mid X'(\tau) \mid \triangle \approx 0$ for all $A\in\mathcal{A}$ such that $\nu(A)\approx 0$.*

Chapter 3

Existence Theorems for ODE's

3.1 Introduction

The aim of this chapter is to present a nonstandard generalization of Carathéodory's Existence Theorem and also a nonstandard proof of Carathéodory's Existence Theorem, which avoid Ascoli's Theorem as well as Lebesgue's Dominated Convergence Theorem.

Throughout this chapter we will suppose that $N \in {}^{\star}\mathbb{N}_{\infty}$ and $\mathbb{T}$ is the hyperfinite time line with respect to the increment $\triangle = \frac{1}{N}$,

$$\mathbb{T} := \{k\triangle : k = 0, 1, 2, \cdots, N-1\} = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1 - \frac{1}{N}\}.$$

$(\mathbb{T}, \mathcal{A}, \nu)$ will be denote the internal counting measure space (see Example 2.48), that is, $\mathcal{A}$ is the set of all internal subsets of $\mathbb{T}$ and

$$\begin{array}{rcl} \nu : \mathcal{A} & \rightarrow & {}^{\star}[0,1] \\ A & \mapsto & \nu(A) = \frac{card(A)}{card(\mathbb{T})}. \end{array}$$

The Loeb space generated by $(\mathbb{T}, \mathcal{A}, \nu)$ will be denoted by $(\mathbb{T}, L(\mathcal{A}), \nu_L)$. As usual, λ will represent the Lebesgue measure and ${}^{\star}\lambda_L$ the Loeb measure generated by ${}^{\star}\lambda$.

If $X : \mathbb{T} \to {}^\star\mathbb{R}$ is an internal function we use the notion X' to denote the nonstandard discrete derivative, that is,

$$\begin{array}{rcl} X' : \mathbb{T} \setminus \{1-\triangle\} & \to & {}^\star\mathbb{R} \\ t & \mapsto & X'(t) := \frac{X(t+\triangle)-X(t)}{\triangle}. \end{array}$$

For classical results in Measure and Integration Theory consult, for example, [Rao87] or [Coh80].

3.2 Nonstandard Carathéodory's Existence Theorem

In this section we present a result about internal functions. As we will see, this result is a generalization of Carathéodory's Existence Theorem since this classical theorem for **ODE's** is a consequence of the first one.

Theorem 3.1 ***(Nonstandard Carathéodory's Existence Theorem)*** [MNa] *Let $F : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}$ be an internal *measurable function. Suppose there exists an internal S-integrable function $M : \mathbb{T} \to {}^\star\mathbb{R}$ such that*

$$\forall (\tau, x) \in \mathbb{T} \times^\star \mathbb{R} \quad | F(\tau, x) | \leq M(\tau).$$

Then, for each $\alpha \in {}^\star\mathbb{R}$, there exists one and only one internal S-absolutely continuous function $X : \mathbb{T} \to {}^\star\mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} X'(\tau) & = & F(\tau, X(\tau)) \quad (\tau \in \mathbb{T} \setminus \{1-\triangle\}) \\ X(0) & = & \alpha \end{array} \right. . \tag{3.1}$$

If α is finite, then $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_{fin}$.

Proof. Define $X : \mathbb{T} \to {}^\star\mathbb{R}$ recursively by

$$\begin{array}{rcl} X(0) & = & \alpha \\ X(t+\triangle) & = & X(t) + F(t, X(t))\triangle \qquad (t \in \mathbb{T} \setminus \{1-\triangle\}). \end{array}$$

X is internal and, by construction, if $t = k\triangle \in \mathbb{T}$,

$$X(t) = \alpha + \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle.$$

Moreover,

$$X'(t) = \frac{X(t+\triangle) - X(t)}{\triangle} = F(t, X(t)) \quad (t \in \mathbb{T} \setminus \{1-\triangle\}).$$

Using the definition of the discrete derivative, it is obvious that there exists only one internal function $X : \mathbb{T} \rightarrow {}^{\star}\mathbb{R}$ such that (3.1) holds.

To prove that X is S-absolutely continuous, we will use Theorem 2.69. Take A an internal subset of $\mathbb{T}$ such that $\nu(A) \approx 0$. Note that

$$\sum_{\tau \in A} \mid X'(\tau) \mid \triangle = \sum_{\tau \in A} \mid F(\tau, X(\tau)) \mid \triangle \leq \sum_{\tau \in A} M(\tau)\triangle = \int_A M d\nu.$$

Since M is S-integrable, $\int_A M d\nu \approx 0$ and therefore

$$\int_A \mid X' \mid d\nu = \sum_{\tau \in A} \mid X'(\tau) \mid \triangle \approx 0$$

which proves that X is S-absolutely continuous.

Finally, note that, for each $\tau = k\triangle \in \mathbb{T}$,

$$\mid X(\tau) - \alpha \mid = \mid \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle \mid \leq \sum_{i=0}^{k-1} M(i\triangle)\triangle \ \leq \int_{\mathbb{T}} M d\nu$$

and $\int_{\mathbb{T}} M d\nu$ is finite since M is S-integrable. Hence, if α is finite, $X(\mathbb{T}) \subseteq {}^{\star}\mathbb{R}_{fin}$. ■

3.3 Carathéodory's Existence Theorem

The next result (see [Rao87, pages 230-240]) gives a characterization of absolutely continuous functions. As usual, $L^1([a,b])$ denotes the set of all Lebesgue measurable functions $f : [a,b] \rightarrow \mathbb{R}$ such that $\int_{[a,b]} \mid f \mid < \infty$.

Theorem 3.2 ***(Fundamental Theorem of Calculus for the Lebesgue integral)*** *A function $f : [a,b] \rightarrow \mathbb{R}$ is absolutely continuous if and only if f is differentiable almost everywhere on $[a,b]$, $f' \in L^1([a,b])$ and*

$$f(x) - f(a) = \int_{[a,x]} f'(t) d\lambda(t) \quad (a \leq x \leq b).$$

Next we will present our nonstandard proof of Carathéodory's Existence Theorem ([MNa]) (a classical proof of this theorem can be found in [CL55, pages 43-44]). In the following, $C([a,b])$ denotes the Banach space of all real continuous functions on $[a,b]$ with the norm defined by $\| f \| := \max_{x\in[a,b]} \mid f(x) \mid$.

Theorem 3.3 ***(Carathéodory's Existence Theorem)*** *Suppose that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, continuous in the second variable and let $x_0 \in \mathbb{R}$. If there exists a Lebesgue integrable function $m : [0,1] \to \mathbb{R}$ such that*

$$\forall (t,x) \in [0,1]\times\mathbb{R} \quad \mid f(t,x) \mid \leq m(t)$$

then there exists $x : [0,1] \to \mathbb{R}$ absolutely continuous such that

$$\begin{cases} x'(t) &= f(t,x(t)) \quad a.a.\ t \in [0,1] \\ x(0) &= x_0 \end{cases} . \tag{3.2}$$

Proof. Since $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function then $F = {}^\star f \mid_{\mathbb{T}\times{}^\star\mathbb{R}} : \mathbb{T}\times{}^\star\mathbb{R} \to {}^\star\mathbb{R}$ is *measurable (with respect to ${}^\star\lambda$). Theorem 2.65 says that ${}^\star m : {}^\star[0,1] \to {}^\star\mathbb{R}$ is S-integrable and therefore $M = {}^\star m \mid_{\mathbb{T}}$ is also S-integrable. Using the Transfer Principle we conclude that

$$\forall (t,x) \in {}^\star[0,1]\times{}^\star\mathbb{R} \ \mid {}^\star f(t,x) \mid \leq \ {}^\star m(t)$$

and then

$$\forall (\tau,x) \in \mathbb{T}\times{}^\star\mathbb{R} \ \mid F(\tau,x) \mid \leq \ M(\tau).$$

Nonstandard Carathéodory's Existence Theorem (Theorem 3.1) shows that there exists an internal S-absolutely continuous $X : \mathbb{T} \to {}^\star\mathbb{R}$ such that

$$\begin{cases} X'(\tau) &= F(\tau, X(\tau)) \quad (\tau \in \mathbb{T}\setminus\{1-\triangle\}) \\ X(0) &= x_0 \end{cases}$$

and for all $\tau = k\triangle \in \mathbb{T}$

$$X(\tau) = x_0 + \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle \in {}^\star\mathbb{R}_{fin}.$$

Since $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_{fin}$, we can choose $r \in \mathbb{R}^+$ such that

$$\forall \tau \in \mathbb{T} \ \mid X(\tau) \mid \leq r. \tag{3.3}$$

Defining $x : [0,1] \to \mathbb{R}$ by

$$x(^\circ\tau) = {}^\circ X(\tau) \qquad (\tau \in \mathbb{T}) \tag{3.4}$$

we conclude, by Theorem 2.35, that x is absolutely continuous. We will prove that this function is a solution to the problem (3.2).

Using the definition and continuity of x we have that

$$X(\tau) \approx x(^\circ\tau) \approx x(\tau) \quad (\tau \in \mathbb{T}).$$

By hypothesis f is Lebesgue measurable, then the function

$$\widetilde{f} : [0,1] \to C([-r,r])$$

defined by

$$\widetilde{f}(t)(z) = f(t,z) \qquad ((t,z) \in [0,1] \times [-r,r])$$

is also Lebesgue measurable. Taking the uniform topology in $C([-r,r])$ and using Anderson's Theorem (Theorem 2.57) we can conclude that

$$^\star\widetilde{f} :\ ^\star[0,1] \to {}^\star C([-r,r])$$

is a lifting of $\widetilde{f}$ with respect to the Loeb measure $^\star\lambda_L$, hence

$$^\star\widetilde{f}(\tau) \approx \widetilde{f}(^\circ\tau) \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0,1].$$

Using the definition of the norm in $C([-r,r])$ we may conclude that

$$\left[\ \forall z \in {}^\star[-r,r]\ {}^\star f(\tau,z) \approx {}^\star f(^\circ\tau,z)\ \right] \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0,1]. \tag{3.5}$$

Since f is continuous in the second variable, we obtain that

$$\left[\ \forall z \in {}^\star[-r,r]\ {}^\star f(\tau,z) \approx f(^\circ\tau,{}^\circ z)\ \right] \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0,1]. \tag{3.6}$$

As X satisfies (3.3) and (3.4), it follows that

$$^\star f(\tau, X(\tau)) \approx f(^\circ\tau, {}^\circ X(\tau)) = f(^\circ\tau, x(^\circ\tau)) \qquad \nu_L - \text{a.a. } \tau \in \mathbb{T}$$

because $\nu_L(\mathbb{T}) = 1$.

Hence,

$$\begin{aligned} G : \mathbb{T} &\to {}^\star\mathbb{R} \\ \tau &\mapsto G(\tau) = {}^\star f(\tau, X(\tau)) \end{aligned}$$

is a lifting of the Lebesgue integrable function

$$\begin{aligned} g : [0,1] &\to \mathbb{R} \\ t &\mapsto g(t) = f(t, x(t)). \end{aligned}$$

Next we will prove that G is S-integrable. Observe that, for all $A \in \mathcal{A}$,

$$\int_A \mid G \mid d\nu \leq \int_A {}^\star m d\nu.$$

Since ${}^\star m : {}^\star[0,1] \to {}^\star\mathbb{R}$ is S-integrable, then

$$\int_{\mathbb{T}} \mid G \mid d\nu \in {}^\star\mathbb{R}_{fin}$$

and

$$\int_A \mid G \mid d\nu \approx 0$$

whenever $\nu(A) \approx 0$, proving that G is S-integrable.

We may now prove that, for all $t \in [0,1]$,

$$x(t) = x_0 + \int_{[0,t]} f(s, x(s)) d\lambda(s).$$

Fix $z \in [0,1]$ and $\tau = k\triangle \in \mathbb{T}$ such that $\tau \approx z$. Observe that

$$\begin{aligned} x(z) &= {}^\circ X(\tau) \\ &= x_0 + {}^\circ\left(\sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle\right) \\ &= x_0 + {}^\circ\left(\sum_{i=0}^{k-1} G(i\triangle)\triangle\right) \\ &= x_0 + \int_{[0,z]} g(t) d\lambda(t) \qquad \text{(Theorem 2.66)} \\ &= x_0 + \int_{[0,z]} f(t, x(t)) d\lambda(t). \end{aligned}$$

Obviously, $x(0) = x_0$. Finally, by Theorem 3.2, we may conclude that x is a solution to the problem (3.2). ∎

3.4 Peano's Existence Theorem

Peano's Existence Theorem is very easily derived from Carathéodory's Existence Theorem.

Theorem 3.4 ***(Peano's Existence Theorem)*** *Suppose $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous and $x_0 \in \mathbb{R}$. Then there exists $x : [0,1] \to \mathbb{R}$ such that*

$$\begin{cases} x'(t) & = & f(t, x(t)) \\ x(0) & = & x_0 \end{cases} . \tag{3.7}$$

Proof. Notice that f satisfies all the conditions of Carathéodory's Existence Theorem (Theorem 3.3). This theorem says that there exists an absolutely continuous function $x : [0,1] \to \mathbb{R}$ that satisfies the integral equation

$$x(z) = x_0 + \int_{[0,z]} f(t, x(t)) d\lambda(t) \quad (z \in [0,1]).$$

Clearly, $x(0) = x_0$. Since f and x are continuous, by the Fundamental Theorem of Calculus (for the Riemann integral), x is such that

$$\forall t \in [0,1] \;\; x'(t) = f(t, x(t)).$$

Therefore, $x : [0,1] \to \mathbb{R}$ satisfies the initial valued problem (3.7). ∎

We included in Appendix A a direct nonstandard proof of Peano's Existence Theorem. The reader may compare the nonstandard proofs of Carathéodory's and Peano's Existence Theorems.

Chapter 4

Palais-Smale conditions

4.1 Introduction

The aim of this chapter is to present nonstandard versions of the Palais-Smale condition (Definition 4.1) and the relations between them. We will see that some of them are generalizations of the classical Palais-Smale condition but still sufficient to prove a new Mountain Pass Theorem (Theorem 5.15).

Suppose $(E, \|\cdot\|)$ is a real Banach space and $f : E \to \mathbb{R}$ is Fréchet differentiable. We will denote by K the set of all critical points of f, that is,

$$K := \{x \in E \;:\; f'(x) = 0\}$$

and, for each $c \in \mathbb{R}$, K_c will denote the set of all critical points with value c, that is,

$$K_c := \{x \in E \;:\; f'(x) = 0 \;\wedge\; f(x) = c\} = K \cap f^{-1}(\{c\}).$$

As usual, $C^1(E, \mathbb{R})$ denotes the set of continuously Fréchet differentiable functionals defined on E.

4.2 The Palais-Smale condition

Many results in Critical Point Theory involve the following condition, originally introduced in 1964 [PS64] by Palais and Smale:

Definition 4.1 *Let E be a real Banach space. We say that $f \in C^1(E,\mathbb{R})$ satisfies the **Palais-Smale condition** ((**PS**) for short) if for all sequence $(u_n)_{n\in\mathbb{N}}$ in E,*

$$(\mathbf{PS}) \qquad (f(u_n))_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to\infty} f'(u_n) = 0$$
$$\Rightarrow \qquad (u_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}$$

The following condition

$$(PS0) \qquad (f(u_n))_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to\infty} f'(u_n) = 0$$
$$\Rightarrow \qquad \exists m \in {}^\star\mathbb{N}_\infty \; u_m \in ns({}^\star E)$$

is a direct translation of (**PS**) in nonstandard terms.

The following is easy to prove.

Proposition 4.2 *Suppose that $f \in C^1(E,\mathbb{R})$ satisfies (**PS**). Then*

1. *For each $a, b \in \mathbb{R}$ such that $a \leq b$,*

   $$\{u \in E \;:\; a \leq f(u) \leq b \;\wedge\; f'(u) = 0\} = f^{-1}([a,b]) \cap K$$

   is a compact set;

2. *If f is bounded, K is a compact set.*

Note that

$$f \text{ satisfies } (\mathbf{PS}) \text{ and } K \text{ is compact } \not\Rightarrow \; f \text{ is bounded}$$

and

$$K \text{ is compact and } f \text{ is bounded } \not\Rightarrow \; f \text{ satisfies } (\mathbf{PS})$$

as can be seen with the following two examples.

Example 4.3 .

1. The real function $f(x) = x^3$ $(x \in \mathbb{R})$ satisfies (**PS**), $K = \{0\}$ is compact and f is not bounded.

2. Let

$$f(x) = \begin{cases} 2 - x^2 & \text{if } x \in [-1, 1] \\ \frac{1}{x^2} & \text{if } x \notin [-1, 1] \end{cases} .$$

   f is bounded, $K = \{0\}$ but f does not satisfies (**PS**).

In Section 4.4, we will present an important class of functionals that satisfy (**PS**). Clearly, the real functions $\exp(x)$, $\cos(x)$, $\sin(x)$ and all the constant functions defined on $\mathbb{R}$ do not satisfy (**PS**).

4.3 Nonstandard Palais-Smale conditions

The following definition often shortens statements.

Definition 4.4 *Suppose $f \in C^1(E, \mathbb{R})$. We say that a sequence $(u_n)_{n\in\mathbb{N}}$ is a **Palais-Smale sequence for** f if $(f(u_n))_{n\in\mathbb{N}}$ is bounded and* $\lim_{n\to\infty} f'(u_n) = 0$.

Therefore,

> *f satisfies the **Palais-Smale condition** (**PS**) if every Palais-Smale sequence for f has a convergent subsequence.*

Suppose $f \in C^1(E, \mathbb{R})$. Next we present some nonstandard variants of (**PS**) (for convenience, we write again $(PS0)$):

$$(PS0) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ \exists m \in {}^\star\mathbb{N}_\infty \ u_m \in ns({}^\star E) \end{array}$$

$$(PS1) \qquad f(u) \in {}^\star\mathbb{R}_{fin} \ \wedge \ f'(u) \approx 0 \Rightarrow u \in ns({}^\star E)$$

$$(PS2) \qquad \begin{array}{c} f(u) \in {}^\star\mathbb{R}_{fin} \ \wedge \ f'(u) \approx 0 \\ \Downarrow \\ u \in fin({}^\star E) \ \wedge \ st(f(u)) \text{ is a critical value of } f \end{array}$$

$$(PS3) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \ \forall n \in {}^\star\mathbb{N}_\infty \ st(f(u_n)) \text{ is a critical value of } f \end{array}$$

$$(PS4) \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \ \exists n \in {}^\star\mathbb{N}_\infty \ st(f(u_n)) \text{ is a critical value of } f \end{array}$$

Proposition 4.5 *If $f \in C^1(E,\mathbb{R})$ then*

$$(PS1) \Leftrightarrow [f(u) \in {}^\star\mathbb{R}_{fin} \ \wedge \ f'(u) \approx 0 \Rightarrow u \in ns({}^\star E) \ \wedge \ st(f(u)) \ \textit{ is a critical value of } \ f].$$

Proof. The implication $\Leftarrow$ is trivial. For the proof of the other implication, suppose that $u \in {}^\star E$ is such that $f(u) \in {}^\star\mathbb{R}_{fin}$ and $f'(u) \approx 0$. By $(PS1)$, there exists $a \in E$ such that $u \approx a$ and, therefore, from the continuity of f and f' it follows that

$$f(a) \approx f(u) \ \text{ and } \ f'(a) \approx f'(u) \approx 0$$

too, so that $f(a) = st(f(u))$ and $f(a)$ is a critical value of f. ∎

Nonstandard versions of (**PS**) and (**PS**) itself are related as follows.

Theorem 4.6 [MNb] *For any real Banach space E we have*

$$(PS1) \Rightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \Rightarrow (PS2) \Leftrightarrow (PS3) \Rightarrow (PS4).$$

Proof. For the equivalence $(\mathbf{PS}) \Leftrightarrow (PS0)$ see Definition 4.1 and following remark. Now, we will prove that $(PS1) \Rightarrow (PS0)$. Let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence. Then, by Proposition 2.23, for all $n \in {}^\star\mathbb{N}_\infty$, $f(u_n) \in {}^\star\mathbb{R}_{fin}$ and $f'(u_n) \approx 0$; hence, by $(PS1)$,

$$\forall n \in {}^\star\mathbb{N}_\infty \; u_n \in ns({}^\star E)$$

and, therefore,

$$\exists m \in {}^\star\mathbb{N}_\infty \; u_m \in ns({}^\star E).$$

The implication $(PS3) \Rightarrow (PS4)$ is clear. Summarizing,

$$(PS1) \Rightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \quad \text{and} \quad (PS3) \Rightarrow (PS4)$$

are true.

Next we will prove that $(PS0) \Rightarrow (PS2)$. Let $u \in {}^\star E$ such that $f(u) \in {}^\star\mathbb{R}_{fin}$ and $f'(u) \approx 0$. Fix $M \in \mathbb{R}^+$ such that $\mid f(u) \mid < M$. Suppose $u \notin fin({}^\star E)$. For each $n \in \mathbb{N}$, define the standard set

$$H_n := \{x \in E \; : \mid f(x) \mid < M \wedge \; \| f'(x) \| < \frac{1}{n} \wedge \; \| x \| > n\}.$$

Since, for each $n \in \mathbb{N}$,

$$u \in {}^\star H_n = \{x \in {}^\star E \; : \mid f(x) \mid < M \wedge \; \| f'(x) \| < \frac{1}{n} \wedge \; \| x \| > n\},$$

we conclude that ${}^\star H_n \neq \emptyset$ and the Transfer Principle says that $H_n \neq \emptyset$. For each $n \in \mathbb{N}$, take $x_n \in H_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence but, for all $n \in {}^\star\mathbb{N}_\infty$, $x_n \notin fin({}^\star E)$, and hence, for all $n \in {}^\star\mathbb{N}_\infty$, $x_n \notin ns({}^\star E)$. This is a contradiction with $(PS0)$ and therefore $u \in fin({}^\star E)$. Now we will prove that if $f(u) \in {}^\star\mathbb{R}_{fin}$ and $f'(u) \approx 0$, then $st(f(u))$ is a critical value of f. Let $\alpha = st(f(u))$ and, for each $n \in \mathbb{N}$, define

$$F_n := \{x \in E \; : \mid f(x) \mid < M \wedge \; \| f'(x) \| < \frac{1}{n} \wedge \; \mid \alpha - f(x) \mid < \frac{1}{n}\}.$$

Since, for each $n \in \mathbb{N}$,

$$u \in {}^\star F_n = \{x \in {}^\star E \; : \mid f(x) \mid < M \wedge \; \| f'(x) \| < \frac{1}{n} \wedge \; \mid \alpha - f(x) \mid < \frac{1}{n}\}$$

then, ${}^{\star}F_n \neq \emptyset$ and therefore $F_n \neq \emptyset$. For each $n \in \mathbb{N}$ take $x_n \in F_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence and from $(PS0)$ we conclude

$$\exists a \in E\ \exists m \in {}^{\star}\mathbb{N}_\infty\ x_m \approx a.$$

Since $f \in C^1(E, \mathbb{R})$,

$$\alpha \approx f(x_m) \approx f(a) \ \text{ and } \ 0 \approx f'(x_m) \approx f'(a).$$

Therefore $\alpha = f(a)$ and $f'(a) = 0$; hence $(PS0) \Rightarrow (PS2)$ is proved.

To prove that $(PS2) \Rightarrow (PS3)$, let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence. Then,

$$\forall n \in {}^{\star}\mathbb{N}_\infty\ [f(u_n) \in {}^{\star}\mathbb{R}_{fin}\ \wedge\ f'(u_n) \approx 0].$$

From $(PS2)$ we conclude that

$$\forall n \in {}^{\star}\mathbb{N}_\infty\ [u_n \in fin({}^{\star}E)\ \wedge\ st(f(u_n)) \text{ is a critical value of } f]$$

thus $(u_n)_{n\in\mathbb{N}}$ is bounded and

$$\forall n \in {}^{\star}\mathbb{N}_\infty\ st(f(u_n)) \text{ is a critical value of } f.$$

Finally we will prove that $(PS3) \Rightarrow (PS2)$. Let $u \in {}^{\star}E$ such that $f(u) \in {}^{\star}\mathbb{R}_{fin}$ and $f'(u) \approx 0$ and $M \in \mathbb{R}^+$ such that $\mid f(u) \mid < M$. If $u \notin fin({}^{\star}E)$, we will be able to construct an unbounded Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ using the sets H_n as in the first part of the proof of $(PS0) \Rightarrow (PS2)$, contradicting $(PS3)$. Hence $u \in fin({}^{\star}E)$. We still need to prove that $\alpha = st(f(u))$ is a critical value of f. As in the second part of the proof of $(PS0) \Rightarrow (PS2)$, we can construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in {}^{\star}\mathbb{N}_\infty$, $f(x_n) \approx \alpha$. From $(PS3)$ we conclude that for all $n \in {}^{\star}\mathbb{N}_\infty$, $st(f(x_n)) = \alpha$ is a critical value of f. ■

More can be said when E is separable:

Theorem 4.7 [MNb] *When E is a separable Banach space,* (**PS**) *and* $(PS1)$ *are equivalent; in other words: if E is a separable Banach space, a C^1 functional $f : E \to \mathbb{R}$ verifies the Palais-Smale condition if and only if almost critical points where f is finite are near-standard.*

Proof. Suppose f satisfies (**PS**) and $u \in {}^{\star}E$ is such that $f(u) \in {}^{\star}\mathbb{R}_{fin}$ and $f'(u) \approx 0$. If $u \notin ns({}^{\star}E)$, it follows from Theorem 2.24 that $u \notin pns({}^{\star}E)$, that is,

$$\exists \epsilon \in \mathbb{R}^+\ \forall y \in E \parallel u - y \parallel > \epsilon.$$

Let $V := \{v_p : p \in \mathbb{N}\}$ be dense in E. We will construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ in E such that for all $N \in {}^\star\mathbb{N}_\infty$, $x_N \notin ns({}^\star E)$ which contradicts (**PS**).

Let $M \in \mathbb{R}^+$ be such that $| f(u) |< M$. For each $n \in \mathbb{N}$, define

$$C_n := \{x \in E \ : | f(x) |< M \wedge \ \| f'(x) \|< \frac{1}{n} \ \wedge \ \forall p \in \mathbb{N} \ [p \leq n \Rightarrow \| x - v_p \| > \epsilon]\}.$$

Since, for each $n \in \mathbb{N}$, $u \in {}^\star C_n$, we conclude that ${}^\star C_n \neq \emptyset$ and therefore $C_n \neq \emptyset$. For each $n \in \mathbb{N}$, take $x_n \in C_n$. Let $N \in {}^\star\mathbb{N}_\infty$ and $v \in E$. Since $\overline{V} = E$, there exists $p_0 \in \mathbb{N}$ such that $\| v - v_{p_0} \|< \frac{\epsilon}{2}$. Since

$$\| x_N - v \| \geq \| x_N - v_{p_0} \| - \| v_{p_0} - v \| > \frac{\epsilon}{2}$$

we conclude that $x_N \notin ns({}^\star E)$ which contradicts (**PS**). ■

Next we show that, in the finite dimensional case, all the nonstandard variants of (**PS**) are equivalent.

Theorem 4.8 [MNb] *If E is finite dimensional*

$$(PS1) \Leftrightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \Leftrightarrow (PS2) \Leftrightarrow (PS3) \Leftrightarrow (PS4).$$

Proof. It follows easily from the last result that $(PS1) \Leftrightarrow$ (**PS**), since all finite dimensional Banach spaces are separable. Then, we may conclude that,

$$(PS1) \Leftrightarrow (\mathbf{PS}) \Leftrightarrow (PS0) \Rightarrow (PS2) \Leftrightarrow (PS3) \Rightarrow (PS4).$$

Therefore, we only need to prove that

$$(PS2) \Rightarrow (PS1) \quad \text{and} \quad (PS4) \Rightarrow (PS0).$$

Since E is finite dimensional, $fin({}^\star E) = ns({}^\star E)$ (Theorem 2.24), and consequently $(PS2) \Rightarrow (PS1)$.

To prove that $(PS4) \Rightarrow (PS0)$, let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence. From $(PS4)$, $(u_n)_{n\in\mathbb{N}}$ is bounded, then

$$\forall n \in {}^\star\mathbb{N}_\infty \ [u_n \in fin({}^\star E) = ns({}^\star E)]$$

and this implies $(PS0)$. ■

Proposition 4.9 *Any C^1 functional in a real Banach space which verifies $(PS4)$ and admits a Palais-Smale sequence, has at least one critical point.*

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for the functional $f \in C^1(E,\mathbb{R})$. Since f verifies $(PS4)$, there exists $m \in {}^\star\mathbb{N}_\infty$ such that $st(f(u_m))$ is a critical value of f. Hence, there exists $a \in E$ such that $f(a) = st(f(u_m))$ and $f'(a) = 0$. ■

Next we present an example which shows that

$$(PS2) \not\Rightarrow (PS1).$$

Example 4.10 Let H be an infinite dimensional real Hilbert space and define

$$\begin{array}{rcl} f : H & \rightarrow & \mathbb{R} \\ x & \mapsto & f(x) = g(\| x \|^2 - 1) \end{array}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$g(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ t^2 \exp^{-\frac{1}{t^2}} & \text{if} \quad t > 0 \end{cases}.$$

Observe that g is a C^1 function and

$$g'(t) \approx 0 \iff [t \leq 0 \ \vee \ t \approx 0]. \tag{4.1}$$

Also,

$$\begin{array}{rcl} h : H & \rightarrow & \mathbb{R} \\ x & \mapsto & \| x \|^2 - 1 \end{array}$$

is a C^1 functional and

$$\forall a \in H \ \forall x \in H \ \langle h'(a), x\rangle = 2a \bullet x$$

where $\cdot \bullet \cdot$ denotes the inner product in H. Therefore, f is a C^1 functional. We will prove that f does not satisfy $(PS1)$ but satisfies $(PS2)$.

By Theorem 2.24 we can take $u \in {}^\star H$ such that $u \in fin({}^\star H) \setminus ns({}^\star H)$ and $\| u \| = 1$. Hence, $f(u) = 0$ and $f'(u) = 0$, which shows that f does not satisfy $(PS1)$.

Note that

$$v \notin fin({}^\star H) \Rightarrow f(v) \notin {}^\star\mathbb{R}_{fin}$$

and

$$f'(v) \approx 0 \ \Leftrightarrow \| f'(v) \| \approx 0 \ \Leftrightarrow \ \forall x \in fin({}^\star H) \ \langle f'(v), x \rangle = (2v \bullet x) g'(\| v \|^2 - 1) \ \approx 0 \quad (4.2)$$

(see the observation after the Definition 2.39).

Next we will prove that

$$f'(v) \approx 0 \ \Rightarrow \ [\| v \| \leq 1 \vee \ \| v \| \approx 1]. \quad (4.3)$$

If $f'(v) \approx 0$ and $v \neq 0$, by (4.2) either $2v \bullet \frac{v}{\|v\|} \approx 0$, and thus $\| v \| \approx 0$, or $g'(\| v \|^2 - 1) \approx 0$, so that, by (4.1), $\| v \| \leq 1$ or $\| v \| \approx 1$; in all possible cases (4.3) holds.

Since

$$[\ \| v \| \leq 1 \vee \ \| v \| \approx 1\ ] \ \Rightarrow \ f(v) \approx 0$$

and 0 is a critical value of f, we may conclude that

$$f(v) \in {}^\star\mathbb{R}_{fin} \ \wedge \ f'(v) \approx 0 \ \Rightarrow \ v \in fin({}^\star H) \ \wedge \ st(f(v)) = 0 \text{ is a critical value of } f$$

proving that f does satisfy $(PS2)$.

Remark 4.11 We present some consequences of Example 4.10*:*

1. If we assume H to be separable, Theorem 4.7 shows that

$$(PS2) \nRightarrow (\mathbf{PS}).$$

For example, take the separable Hilbert space

$$H = l^2 := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ : \ \sum_{n\in\mathbb{N}} | \ x_n \ |^2 < +\infty \right\}$$

where, for each $(x_n)_{n\in\mathbb{N}} \in l^2$,

$$\| (x_n)_{n\in\mathbb{N}} \| := \left(\sum_{n\in\mathbb{N}} | \ x_n \ |^2 \right)^{\frac{1}{2}}.$$

Let $(u_j)_{j\in\mathbb{N}}$ be a sequence in l^2 such that, for each $j \in \mathbb{N}$,

$$u_j = (u_n)_{n\in\mathbb{N}} = \begin{cases} 0 & \text{if} \quad n \neq j \\ \frac{1}{2} & \text{if} \quad n = j \end{cases}.$$

Note that, for each $j \in \mathbb{N}$, $\| u_j \| = \frac{1}{2}$, therefore,

$$\forall j \in \mathbb{N} \; [\; f(u_j) = 0 \;\wedge\; f'(u_j) = 0 \;],$$

where f is the functional defined on Example 4.10.

Hence, $(u_j)_{j\in\mathbb{N}}$ is a Palais-Smale sequence. Since $(u_j)_{j\in\mathbb{N}}$ does not have a convergent subsequence, f does not satisfy (**PS**).

2. The fact that $f \in C^1(E, \mathbb{R})$ and satisfies $(PS2)$, does not imply that the sets

$$f^{-1}([a,b]) \cap K \qquad (a \leq b)$$

are compact (see Proposition 4.2).

We notice that, by Proposition 2.23, condition $(PS3)$ is equivalent to

$$\begin{gathered} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \text{ all convergent subsequences of } (f(u_n))_{n\in\mathbb{N}} \\ \text{converge to a critical value of } f \end{gathered}$$

and $(PS4)$ is equivalent to

$$\begin{gathered} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence} \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \text{ there is a subsequence of } (f(u_n))_{n\in\mathbb{N}} \\ \text{which converges to a critical value of } f. \end{gathered}$$

Moreover, in the finite dimensional case, these two standard conditions are equivalent.

Remark 4.12 In [AE84] Aubin and Ekeland introduced the following form of the Palais-Smale condition:

$f \in C^1(E, \mathbb{R})$ satisfies (WPS) condition on $\Omega \subseteq E$ if for every Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ in Ω, there exists $\overline{x} \in E$ such that

$$\liminf_{n\to\infty} f(u_n) \leq f(\overline{x}) \leq \limsup_{n\to\infty} f(u_n) \quad \textit{and} \quad f'(\overline{x}) = 0.$$

In fact, $(PS4)$ condition implies (WPS) on Ω for any $\Omega \subseteq E$. To prove this implication, suppose that $f \in C^1(E,\mathbb{R})$ satisfies $(PS4)$ and $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence in Ω. Then, there exists a subsequence $(f(u_{\sigma(n)}))_{n\in\mathbb{N}}$ of $(f(u_n))_{n\in\mathbb{N}}$ which converges to a critical value of f; hence, there exists $\overline{x} \in E$ such that

$$\lim_{n\to\infty} f(u_{\sigma(n)}) = f(\overline{x}) \qquad \wedge \qquad f'(\overline{x}) = 0.$$

Since

$$\liminf_{n\to\infty} f(x_n) \le \lim_{n\to\infty} f(u_{\sigma(n)}) \le \limsup_{n\to\infty} f(x_n),$$

then f satisfies (WPS) condition on Ω.

4.4 Coercive functionals

Definition 4.13 *The functional $f : E \to \mathbb{R}$ is called* ***coercive*** *if* $\lim_{\|x\|\to+\infty} f(x) = +\infty$.

In the following, if $b \notin {}^\star\mathbb{R}_{fin}$ and $b > 0$ we will write $b \approx +\infty$; $b \approx -\infty$ means that $b \notin {}^\star\mathbb{R}_{fin}$ and $b < 0$.

It is easily seen that

$$\lim_{\|x\|\to+\infty} f(x) = +\infty \;\Leftrightarrow\; \forall x \in {}^\star E \;[\; \| x \| \approx +\infty \;\Rightarrow\; f(x) \approx +\infty \;];$$

therefore

$$f : E \to \mathbb{R} \textit{ is coercive if and only if } \forall x \in {}^\star E \;[\; \| x \| \approx +\infty \;\Rightarrow\; f(x) \approx +\infty \;].$$

The following is a known result.

Theorem 4.14 *If E is a finite dimensional real Banach space and $f : E \to \mathbb{R}$ is continuous and coercive, then f has a minimum on E.*

Proof. By the Discretization Principle (Theorem 2.15), there exists an hyperfinite set D such that $E \subseteq D \subseteq {}^\star E$. Therefore $f_{|_D} : D \to {}^\star\mathbb{R}$ has a minimum $m \in {}^\star\mathbb{R}$. Since f is coercive, $m = f(x_0)$ for some $x_0 \in D \cap fin({}^\star E)$. Since E is finite dimensional, $ns({}^\star E) = fin({}^\star E)$

(Theorem 2.24) and so $x_0 \in D \cap ns(^{\star}E)$. Consequently, there exists $a \in E$ such that $x_0 \approx a$ and, since f is continuous, we have $f(a) \approx f(x_0)$ (Theorem 2.28), so that

$$st(m) \ = \ st(f(x_0)) \ = f(a);$$

but $f(x_0) = m \leq f(x)$, for all $x \in E$, because $E \subseteq D$, and thus

$$f(a) \ = \ st(m) \ \leq \ st(f(x)) \ = f(x),$$

for all $x \in E$, and $f(a)$ is indeed the minimum of f. ■

If E is an arbitrary real Banach space the conclusion of Theorem 4.14 is false, as can be seen with the following examples.

Example 4.15 Consider the real Banach space

$$\chi = \{u \in C([0,1], \mathbb{R}) : u(0) = u(1) = 0\}$$

with the norm

$$\| u \| := \max_{t \in [0,1]} | u(t) | .$$

Let $I : \chi \to \mathbb{R}$ be such that

$$I(u) = \int_0^1 (u(x) + 1)^2 dx.$$

Clearly, I is continuous, coercive and $\inf_{u \in \chi} I(u) \geq 0$.

Now we will prove that $\inf_{u \in \chi} I(u) = 0$. For each $n \in \mathbb{N}$ and $n \geq 2$ define

$$u_n(x) = \begin{cases} -2^n x & \text{if} \quad x \in [0, \frac{1}{2^n}[\\ -1 & \text{if} \quad x \in [\frac{1}{2^n}, 1 - \frac{1}{2^n}[\\ 2^n(x-1) & \text{if} \quad x \in [1 - \frac{1}{2^n}, 1] \end{cases} .$$

Since each $u_n \in \chi$ and

$$\lim_{n \to +\infty} I(u_n) = \lim_{n \to +\infty} \int_0^1 (u_n(x) + 1)^2 (x) dx = 0,$$

we may conclude that

$$\inf_{u \in \chi} I(u) = 0.$$

However, I has no minimum because $I(u) = 0$ if and only if $u(x) = -1$ for all $x \in]0,1[$, and there is no a continuous function which satisfies this condition and $u(0) = u(1) = 0$.

Now we present a coercive continuous functional that is not even bounded from below.

Example 4.16 Denote by l^∞ the set of all bounded sequences of real numbers. Recall that l^∞ is a real Banach space with the norm

$$\| x \| := \sup_{i \in \mathbb{N}} | x_i |,$$

where $x = (x_i)_{i \in \mathbb{N}}$.

Define $\mathcal{B} = \{e_i : i \in \mathbb{N}\}$ where, for each $i \in \mathbb{N}$, e_i is the sequence that in the position i is equal to 1 and is 0 in the other positions. Observe that $\mathcal{B}$ is a closed set. Define on $\mathcal{B}$ the continuous function

$$F_1 : \mathcal{B} \to \mathbb{R} \quad \text{where} \quad F_1(e_i) = -i.$$

From the Tietze-Urysohn Theorem (also called Tietze Extension Theorem) (see [Lan93, page 42]), there exists a continuous function that extends F_1; denote this function $\widehat{F}_1 : l^\infty \to \mathbb{R}$.

Take

$$F_2 : l^\infty \to \mathbb{R} \quad \text{such that} \quad F_2(x) = \| x \| .$$

By Urysohn's Theorem (see [Lan93, page 40]), there exists a continuous function $g : \mathbb{R} \to [0, 1]$ such that

$$g(x) = \begin{cases} 1 & \text{if} \quad | x | \leq 1 \ \vee \ | x | \geq 4 \\ 0 & \text{if} \quad 2 \leq | x | \leq 3 \\ \in [0,1] & \text{if} \quad 1 < | x | < 2 \ \vee \ 3 < | x | < 4 \end{cases} .$$

If we define $G : l^\infty \to \mathbb{R}$ by

$$G(x) = \begin{cases} g(\| x \|)\widehat{F}_1(x) & \text{if} \quad \| x \| < \frac{5}{2} \\ g(\| x \|)F_2(x) & \text{if} \quad \| x \| \geq \frac{5}{2} \end{cases}$$

we obtain a continuous functional that is coercive. Nevertheless, G is not bounded from below because

$$\lim_{i \to +\infty} G(e_i) = \lim_{i \to +\infty} g(\| e_i \|)\widehat{F}_1(e_i) = \lim_{i \to +\infty} F_1(e_i) = -\infty.$$

The next result gives us another nonstandard characterization for coercive functionals.

Proposition 4.17 *A functional $f : E \to \mathbb{R}$ is coercive if and only if*

$$\forall x \in {}^\star E\ [\ (\ f(x) \in {}^\star\mathbb{R}_{fin}\ \vee\ f(x) < 0\ )\ \Rightarrow\ x \in fin({}^\star E)\ ]. \tag{4.4}$$

Proof. Suppose f is coercive and $x \in {}^\star E$ is such that $f(x) \in {}^\star\mathbb{R}_{fin}$ or $f(x) < 0$. Then, x must be finite, otherwise, since f is coercive, $f(x) \approx +\infty$, which is a contradiction.

Suppose now that f satisfies (4.4). We must prove that f is coercive; let $x \in {}^\star E$ be such that $\| x \| \approx +\infty$. Since $x \notin fin({}^\star E)$, by (4.4),

$$f(x) \notin {}^\star\mathbb{R}_{fin}\ \wedge\ f(x) \geq 0$$

and this means that $f(x) \approx +\infty$; hence, f is coercive. ∎

In order to show that the implication in (4.4) cannot be an equivalence, we present a coercive function $f : \mathbb{R} \to \mathbb{R}$ for which

$$\exists x \in {}^\star E\ [\ x \in fin({}^\star E)\ \wedge\ f(x) \geq 0\ \wedge\ f(x) \notin {}^\star\mathbb{R}_{fin}\ ].$$

Example 4.18 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \in [-1,1] \setminus \{0\} \\ 0 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad x \notin [-1,1] \end{cases}.$$

Observe that f is coercive and, for all $0 \neq \epsilon \approx 0$, $f(\epsilon) \approx +\infty$.

If we introduce extra conditions to f we have the following result.

Proposition 4.19 *Let E be a finite dimensional real Banach space and $f : E \to \mathbb{R}$ a continuous map. The following conditions are equivalent:*

1. *f is coercive;*
2. *$\forall x \in {}^\star E\ [\ (\ f(x) \in {}^\star\mathbb{R}_{fin}\ \vee\ f(x) < 0\ )\ \Leftrightarrow\ x \in fin({}^\star E)\ ]$;*
3. *$\forall x \in {}^\star E\ [\ f(x) \approx +\infty \Leftrightarrow x \notin fin({}^\star E)\ ]$.*

Proof. Obviously *2.* $\Leftrightarrow$ *3.* $\Rightarrow$ *1.* Lets prove that *1.* $\Rightarrow$ *2.* Suppose that f is coercive; using Proposition 4.17, it remains to prove that

$$\forall x \in {}^\star E \; [\; x \in fin({}^\star E) \;\Rightarrow\; (\; f(x) \in {}^\star\mathbb{R}_{fin} \;\vee\; f(x) < 0 \;)\;].$$

Let $x \in fin({}^\star E)$; since E is finite dimensional, $x \in ns({}^\star E)$ and the continuity of f implies that $f(x) \in ns({}^\star\mathbb{R}) = {}^\star\mathbb{R}_{fin}$. ■

Proposition 4.20 *Suppose that E is a finite dimensional real Banach space and $f : E \to \mathbb{R}$ is continuous. If f is coercive then*

$$\forall x \in {}^\star E \; [\; f(x) \in {}^\star\mathbb{R}_{fin} \;\Leftrightarrow\; x \in ns({}^\star E) \;]. \tag{4.5}$$

Proof. Suppose that E is finite dimensional and $f : E \to \mathbb{R}$ is continuous and coercive. From Proposition 4.19 it is obvious that

$$\forall x \in {}^\star E \; [\; f(x) \in {}^\star\mathbb{R}_{fin} \;\Rightarrow\; x \in ns({}^\star E) \;].$$

Condition

$$\forall x \in {}^\star E \; [\; x \in ns({}^\star E) \;\Rightarrow\; f(x) \in {}^\star\mathbb{R}_{fin} \;]$$

follows easily from the continuity of f. ■

Remark 4.21 Notice that, even when f is continuous,

$$(4.5) \not\Rightarrow f \;\text{ is coercive.}$$

For example, $f(x) = x^3 \;\; (x \in \mathbb{R})$ is continuous, satisfies (4.5) but is not coercive.

If E is finite dimensional, $f : E \to \mathbb{R}$ is continuous and bounded from below, we have a nicer characterization of coercive functionals:

Proposition 4.22 *Let E be a finite dimensional real Banach space and $f : E \to \mathbb{R}$ be continuous and bounded from below. Then, f is coercive if and only if*

$$\forall x \in {}^\star E \; [\; f(x) \in {}^\star\mathbb{R}_{fin} \;\Leftrightarrow\; x \in ns({}^\star E) \;].$$

Proof. Let E be finite dimensional and $f : E \to \mathbb{R}$ be continuous and bounded from below. Using Proposition 4.20 it remains to prove that

$$\big(\forall x \in {}^\star E \; [\; f(x) \in {}^\star\mathbb{R}_{fin} \;\Leftrightarrow\; x \in ns({}^\star E) \;]\big) \Rightarrow \; f \; \text{ is coercive.}$$

Let $x \in {}^\star E$ be such that $\| \; x \; \| \approx +\infty$. Then $x \notin fin({}^\star E) = ns({}^\star E)$ and, by hypothesis, $f(x) \notin {}^\star\mathbb{R}_{fin}$. Since f is bounded from below, $f(x) \approx +\infty$. Thus, f is coercive. ∎

Remark 4.23 Example 4.18 also shows that if we remove the continuity of the functional, then the conclusion of Proposition 4.22 is false.

Next we will prove that in the finite dimensional case, coercivity is a stronger condition than $(PS1)$.

Proposition 4.24 *If E is a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$ is coercive, then f satisfies $(PS1)$.*

Proof. Let $u \in {}^\star E$ be such that $f(u) \in {}^\star\mathbb{R}_{fin}$ and $f'(u) \approx 0$. We must prove that $u \in ns({}^\star E)$. Since f is bounded from below (see Theorem 4.14), the conclusion follows from Proposition 4.22. ∎

Since in the finite dimensional case, (**PS**) and $(PS1)$ are equivalent, a known proposition follows:

Proposition 4.25 *If E is a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$ is coercive, then f satisfies* (**PS**).

4.5 Palais-Smale conditions *per* level

In the following we present a weaker *compactness condition* for C^1 functionals introduced in 1980 [BCN80] by Brézis, Coron and Nirenberg. In the survey books [GT01] and [Jab03] the reader can find more variants of the (**PS**) condition.

Definition 4.26 *Suppose $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We say that $(u_n)_{n\in\mathbb{N}}$ is a* ***Palais-Smale sequence of level*** c *(for f) if*

$$\lim_{n\to\infty} f(u_n) = c \quad \textit{and} \quad \lim_{n\to\infty} f'(u_n) = 0.$$

f sati sfies the ***Palais-Smale condition of level*** c*,* $(\mathbf{PS})_c$*, if every Palais-Smale sequence of level c has a convergent subsequence.*

Remark 4.27 Suppose that $f \in C^1(E, \mathbb{R})$. Then

1. f satisfies $(\mathbf{PS})$ if and only if f satisfies $(\mathbf{PS})_c$ for all $c \in \mathbb{R}$;

2. If f satisfies $(\mathbf{PS})_c$, then the set of critical points of value c,

$$K_c = \{x \in E \ : \ f'(x) = 0 \ \wedge \ f(x) = c\}$$

is compact.

Example 4.28 The function $\exp(x) : \mathbb{R} \to \mathbb{R}$ satisfies $(\mathbf{PS})_c$ for all c except for $c = 0$. The real functions $\sin(x)$ and $\cos(x)$ defined in $\mathbb{R}$ satisfy $(\mathbf{PS})_c$ for all c except for $c = 1$ and $c = -1$.

4.6 Nonstandard Variants of Palais-Smale conditions *per* level

As above, E is a real Banach space, $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$.

$$(PS0)_c \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ \exists m \in {}^\star\mathbb{N}_\infty \ u_m \in ns({}^\star E) \end{array}$$

$$(PS1)_c \qquad f(u) \approx c \ \wedge \ f'(u) \approx 0 \Rightarrow u \in ns({}^\star E)$$

$$(PS2)_c \qquad \begin{array}{c} f(u) \approx c \ \wedge \ f'(u) \approx 0 \\ \Downarrow \\ u \in fin(^{\star}E) \ \wedge \ st(f(u)) \text{ is a critical value of } f \end{array}$$

$$(PS3)_c \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \ \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded} \wedge \forall n \in {}^{\star}\mathbb{N}_\infty \ st(f(u_n)) \text{ is a critical value of } f \end{array}$$

$$(PS4)_c \qquad \begin{array}{c} (u_n)_{n\in\mathbb{N}} \ \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \text{ is bounded } \wedge \ \exists n \in {}^{\star}\mathbb{N}_\infty \ st(f(u_n)) \text{ is a critical value of } f \end{array}$$

Remark 4.29 We can clearly see that

- For each $i = 1, 2, 3, 4$, f satisfies (PSi) if and only if f satisfies $(PSi)_c$ for all $c \in \mathbb{R}$;
- f satisfies $(PS1)_c$ if and only if

$$f(u) \approx c \ \wedge \ f'(u) \approx 0 \Rightarrow u \in ns(^{\star}E) \ \wedge \ c \ \text{ is a critical value of } f;$$

- f satisfies $(PS2)_c$ if and only if

$$f(u) \approx c \ \wedge \ f'(u) \approx 0 \Rightarrow u \in fin(^{\star}E) \ \wedge \ c \ \text{ is a critical value of } f;$$

- if $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence of level c then,

$$\forall n \in {}^{\star}\mathbb{N}_\infty \ st(f(u_n)) = c$$

  hence, $(PS3)_c \Leftrightarrow (PS4)_c$ and we can simply say that f satisfies $(PS3)_c$ and $(PS4)_c$ if and only if

$$\begin{array}{c} (u_n)_{n\in\mathbb{N}} \ \text{ is a Palais-Smale sequence of level } c \\ \Downarrow \\ (u_n)_{n\in\mathbb{N}} \ \text{ is bounded } \wedge \ c \text{ is a critical value of } f. \end{array}$$

The following result establish the relations between the classical $(\mathbf{PS})_c$ condition with the nonstandard variants defined above.

Theorem 4.30 [MNb] *For any real Banach space E we have*

$$(PS1)_c \Rightarrow (\mathbf{PS})_c \Leftrightarrow (PS0)_c \Rightarrow (PS2)_c \Leftrightarrow (PS3)_c \Leftrightarrow (PS4)_c.$$

Proof. Clearly

$$(PS1)_c \Rightarrow (\mathbf{PS})_c \Leftrightarrow (PS0)_c \quad \text{and} \quad (PS3)_c \Leftrightarrow (PS4)_c.$$

For the proof of $(PS0)_c \Rightarrow (PS2)_c$ take $u \in {}^\star E$ such that $f(u) \approx c$ and $f'(u) \approx 0$. Suppose $u \notin fin({}^\star E)$ and for each $n \in \mathbb{N}$, let

$$Y_n := \{x \in E : \mid f(x) - c \mid < \frac{1}{n} \wedge \ \| f'(x) \| < \frac{1}{n} \wedge \ \| x \| > n\}.$$

Using the same arguments used in the proof $(PS0) \Rightarrow (PS2)$ (Theorem 4.6) we can construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ of level c such that, for all $n \in {}^\star\mathbb{N}_\infty$, $x_n \notin ns({}^\star E)$, which contradicts $(PS0)_c$; therefore, u must be finite. It remains to show that if $f(u) \approx c$ and $f'(u) \approx 0$, then c is a critical value of f. For each $n \in \mathbb{N}$ take

$$X_n := \{x \in E : \mid f(x) - c \mid < \frac{1}{n} \wedge \ \| f'(x) \| < \frac{1}{n}\}$$

and repeat the argument used in the proof $(PS0) \Rightarrow (PS2)$.

The proof $(PS2)_c \Leftrightarrow (PS3)_c$ is analogous to the proof $(PS2) \Leftrightarrow (PS3)$. ∎

Theorem 4.31 [MNb] *Let E be a real separable Banach space and $f \in C^1(E, \mathbb{R})$. Then f satisfies $(\mathbf{PS})_c$ if and only if f satisfies $(PS1)_c$.*

Proof. Suppose f satisfies $(\mathbf{PS})_c$ and $u \in {}^\star E$ is such that $f(u) \approx c$ and $f'(u) \approx 0$. Choose $V := \{v_p : p \in \mathbb{N}\}$ dense in E and suppose that $u \notin ns({}^\star E)$. For each $n \in \mathbb{N}$, define

$$G_n := \{x \in E \ : \mid f(x) - c \mid < \frac{1}{n} \wedge \ \| f'(x) \| < \frac{1}{n} \ \wedge \ \forall p \in \mathbb{N} \ [p \leq n \Rightarrow \| x - v_p \| > \epsilon]\}.$$

In a similar way as we did in the proof of Theorem 4.7, we can construct a Palais-Smale sequence $(x_n)_{n\in\mathbb{N}}$ of level c such that, for all $n \in {}^\star\mathbb{N}_\infty$, $x_N \notin ns({}^\star E)$; this is a contradiction with $(PS0)_c$. ∎

The variant of Theorem 4.8 can be easily proved.

Theorem 4.32 [MNb] *If E has finite dimension, then*

$$(PS1)_c \Leftrightarrow (\mathbf{PS})_c \Leftrightarrow (PS0)_c \Leftrightarrow (PS2)_c \Leftrightarrow (PS3)_c \Leftrightarrow (PS4)_c.$$

Remark 4.33 Example 4.10 also shows that condition $(PS2)_c$ does generalize $(\mathbf{PS})_c$ and $(PS1)_c$ when $c = 0$.

Chapter 5

Mountain Pass Theorems

5.1 Introduction

The purpose of this chapter is to present

- generalizations of classical results about C^1 functionals bounded from below (Corollary 5.4 and Proposition 5.8);
- a Mountain Pass Theorem with a nonstandard Palais-Smale condition (Theorem 5.15) which is a generalization of classical Mountain Pass Theorems;
- a nonstandard proof of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for coercive functionals defined in finite dimensional real Banach spaces;
- a new Mountain Pass Theorem in finite dimension without Palais-Smale conditions (Theorem 5.25);
- a new Mountain Pass Theorem without Palais-Smale conditions for functionals defined in real Hilbert spaces (Theorem 5.29).

In this chapter we will suppose that $(E, \|\cdot\|)$ is a real Banach space and $f \in C^1(E, \mathbb{R})$. If $a \in E$ and $r \in \mathbb{R}^+$, we will use the notations $\mathbf{B}_r(a)$ and $\overline{\mathbf{B}}_r(a)$ to denote, respectively, the open ball and the closed ball centered at a and radius r. Specified references are sources for proofs.

5.2 Critical Points and the Quantitative Deformation Lemma

In the next section we will present two important results in Critical Point Theory: the **Mountain Pass Theorem of Ambrosetti-Rabinowitz** (Theorem 5.10) and one of its generalizations, the **Mountain Pass Theorem of Brézis-Coron-Nirenberg** (Theorem 5.14). Proofs of these theorems use some form of the Deformation Lemma, a very technical lemma that involves the concept of pseudo-gradient vector field (see [Jab03], [GT01], [Maw02] or [Rab86]); it can also be proved using Ekeland's Variational Principle and Von Neumann min-max Theorem (see [Eke79] and [Fig88]).

The *deformation technique* was introduced in 1934 by Lusternik and Schnirelman [LS34] and consists of deforming a given C^1 functional outside the set of critical points. In 1983 Willem proved in [Wil83] the Quantitative Deformation Lemma (see also [Maw02, pages 12-14] or [Jab03, pages 38-40]) without the (**PS**) condition; the usual Deformation Lemma (see [Rab86, pages 82-85]) is proved for functionals that satisfy the (**PS**) condition.

For $S \subseteq E$, $\alpha \in \mathbb{R}^+$ and $c \in \mathbb{R}$, we use the following notations

$$f^c := \{x \in E \ : \ f(x) \leq c\} \quad \text{and} \quad S_\alpha := \{x \in E \ : \ dist(x, S) \leq \alpha\}$$

where

$$dist(x, S) := \inf\{\| x - y \| : \ y \in S\}.$$

Let us introduce more notation

$$C([0,1] \times E, E) := \{\eta : [0,1] \times E \to E \ : \ \eta \text{ is continuous}\}$$

and

$$C([0,1], E) := \{\gamma : [0,1] \to E \ : \ \gamma \text{ is continuous}\}.$$

For our purposes, the following version of the **Quantitative Deformation Lemma** is sufficient.

Lemma 5.1 *Let $f \in C^1(E, \mathbb{R})$, $S \subseteq E$, $c \in \mathbb{R}$, $\epsilon, \delta \in \mathbb{R}^+$ be such that*

$$\forall y \in f^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta} \quad \| f'(y) \| \geq \frac{8\epsilon}{\delta}.$$

Then there exists $\eta \in C([0,1] \times E, E)$ such that

1. $\eta(t, y) = y$ *if* $y \notin f^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta}$;
2. $\eta(1, f^{c+\epsilon} \cap S) \subseteq f^{c-\epsilon}$.

Lemma 5.1 proves the following variant of **Ekeland's Variational Principle** [Eke79]:

Theorem 5.2 [Maw02, page 14] *Suppose $f \in C^1(E, \mathbb{R})$ is bounded from below. Let $\epsilon \in \mathbb{R}^+$ and $z \in E$ be such that*

$$f(z) \leq \inf_{x \in E} f(x) + \epsilon.$$

Then, for any $\delta \in \mathbb{R}^+$, there exists $u \in E$ with the following properties:

1. $f(u) \leq \inf_{x \in E} f(x) + 2\epsilon$;
2. $\| u - z \| \leq 2\delta$;
3. $\| f'(u) \| < \frac{8\epsilon}{\delta}$.

Proof. Take $S = \{z\}$ and $c = \inf_{x \in E} f(x)$. If we suppose that exists $\delta \in \mathbb{R}^+$ such that

$$\forall y \in f^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta} \quad \| f'(y) \| \geq \frac{8\epsilon}{\delta}$$

then, by condition *2.* of Lemma 5.1, there exists $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, f^{c+\epsilon} \cap S) \subseteq f^{c-\epsilon}.$$

Therefore

$$f(\eta(1, z)) \leq c - \epsilon,$$

which contradicts the definition of c. ■

The last theorem shows the following existence result.

Corollary 5.3 *Let $f \in C^1(E, \mathbb{R})$ be bounded from below. Then, there exists $u \in^\star E$ such that*

$$f(u) \approx \inf_{x \in E} f(x) \quad \textit{and} \quad f'(u) \approx 0.$$

Proof. The result follows applying Transfer Principle to Theorem 5.2 and taking $0 < \epsilon \approx 0$ and $\delta = \sqrt{\epsilon}$. ■

Clearly, a C^1 functional which is bounded from below needs not to have a minimum; a classical example is the exponential function. But, if the functional satisfies some kind of *compactness*, then it has a minimum:

Corollary 5.4 *Let $f \in C^1(E, \mathbb{R})$ be bounded from below. If f satisfies $(PS2)_c$ for $c = \inf_{x \in E} f(x)$, then c is a minimum of f.*

Proof. By Corollary 5.3 there exists $u \in {}^\star E$ such that

$$f(u) \approx \inf_{x \in E} f(x) = c \quad \text{and} \quad f'(u) \approx 0.$$

Since f satisfies $(PS2)_c$, c is a critical value of f; therefore, c is a minimum of f. ■

The following result is an easy consequence of the previous corollary.

Corollary 5.5 *Let $f \in C^1(E, \mathbb{R})$ be bounded from above. If f satisfies $(PS2)_c$ for $c = \sup_{x \in E} f(x)$, then c is a maximum of f.*

Proof. Apply Corollary 5.4 to $-f$. ■

We recall that, for any real Banach space, we have

$$(\mathbf{PS})_c \Rightarrow (PS2)_c$$

and

$$(PS2)_c \nRightarrow (\mathbf{PS})_c$$

hence, Corollary 5.4 and Corollary 5.5 are generalizations of the classical results:

Corollary 5.6 *Let $f \in C^1(E, \mathbb{R})$ be bounded from below. If f satisfies $(\mathbf{PS})_c$ for $c = \inf_{x \in E} f(x)$, then c is a minimum of f.*

Corollary 5.7 *Let $f \in C^1(E, \mathbb{R})$ be bounded from above. If f satisfies $(\mathbf{PS})_c$ for $c = \sup_{x \in E} f(x)$, then c is a maximum of f.*

If $f : E \to \mathbb{R}$ is Fréchet differentiable and $f(a)$ is an extreme value, then a is a critical point of f; would it be true that

> *When $\mathcal{H}$ is a hyperfinite set such that $E \subseteq \mathcal{H} \subseteq {}^\star E$ and $f(\omega) = \min_{x\in\mathcal{H}} f(x)$ or $f(\omega) = \max_{x\in\mathcal{H}} f(x)$, then ω is an almost critical point of f?*

The answer is no: the function

$$f(x) = \exp(x) + x \qquad (x \in \mathbb{R})$$

has a minimum and a maximum on any hyperfinite set $\mathcal{H}$ satisfying $\mathbb{R} \subseteq \mathcal{H} \subseteq {}^\star\mathbb{R}$, but $f'(x) = \exp(x) + 1 \not\approx 0$ therefore, f has no almost critical points.

From Proposition 4.24 and Theorem 4.8 it follows that

> *If E is a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$ is coercive, then f satisfies $(PS4)$.*

The converse of this result is, in general, not true. For example, the function $f(x) = x^3$ $(x \in \mathbb{R})$ satisfies $(PS4)$ but f is not coercive. The next result shows that $(PS4)$ for functionals bounded from below is a stronger condition than coercivity. Our proof is similar to the one presented in [MM02, page 5].

Proposition 5.8 *Let $f \in C^1(E, \mathbb{R})$ bounded from below. If f satisfies $(PS4)$, then f is coercive.*

Proof. Suppose that f is not coercive. Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\| x_n \| \to +\infty$ and $(f(x_n))_{n\in\mathbb{N}}$ is bounded. Let $c = \inf_{x\in E} f(x)$ and, for each $n \in \mathbb{N}$, take $\epsilon_n = f(x_n) - c + \frac{1}{n}$ and $\delta_n = \frac{1}{4} \| x_n \|$. By Theorem 5.2 there exists, for each $n \in \mathbb{N}$, $u_n \in E$ such that

$$f(u_n) \leq 2f(x_n) - c + \frac{2}{n} \quad \wedge \quad \| u_n - x_n \| \leq \frac{1}{2} \| x_n \| \quad \wedge \quad \| f'(u_n) \| < \frac{32(f(x_n) - c + \frac{1}{n})}{\| x_n \|}.$$

Thus, $(f(u_n))_{n\in\mathbb{N}}$ is bounded and $f'(u_n) \to 0$. Since, for each $n \in \mathbb{N}$,

$$\| u_n \| \geq \| x_n \| - \| u_n - x_n \| \geq \frac{1}{2} \| x_n \|$$

$(u_n)_{n\in\mathbb{N}}$ is not bounded, contradicting $(PS4)$. ∎

Recall that, for any real Banach space,

$$(\mathbf{PS}) \Rightarrow (PS4)$$

and

$$(PS4) \nRightarrow (\mathbf{PS}),$$

hence Proposition 5.8 generalizes the following result.

Proposition 5.9 [CLW90] *Let $f \in C^1(E,\mathbb{R})$ be bounded from below. If f satisfies $(\mathbf{PS})_c$ condition for all $c \in \mathbb{R}$, then f is coercive.*

5.3 Mountain Pass Theorems in arbitrary real Banach spaces

We begin this section presenting the Mountain Pass Theorem of Ambrosetti-Rabinowitz proved in 1973 [AR73] and frequently used to prove the existence of nontrivial solutions of nonlinear problems.

Theorem 5.10 ***(Mountain Pass Theorem of Ambrosetti-Rabinowitz)*** *Let E be a real Banach space and $f \in C^1(E,\mathbb{R})$. Suppose that*

1. *there exist $x_1, x_2 \in E$ and $r \in \mathbb{R}^+$ such that $\| x_1 - x_2 \| > r$ and*
$$k_0 := \max\{f(x_1), f(x_2)\} \; < \inf_{\|y-x_1\|=r} f(y);$$
2. *$\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;*
3. *f satisfies $(\mathbf{PS})$.*

Then $k_1 > k_0$ and k_1 is a critical value of f.

Condition *1.* of Theorem 5.10 will be used several times in this work therefore, in order to shorten statements, we will present the following definition:

Definition 5.11 *Let $f \in C(E, \mathbb{R})$ and $x_1, x_2 \in E$. We say that f satisfies the* ***mountain pass geometry with respect to*** x_1 ***and*** x_2 *if there exists $r \in \mathbb{R}^+$ such that*

$$\| x_1 - x_2 \| > r \quad \textit{and} \quad \max\{f(x_1), f(x_2)\} < \inf_{\|y-x_1\|=r} f(y).$$

The following result justifies the purpose of the mountain pass geometry.

Proposition 5.12 *Let E be a real Banach space, $x_1, x_2 \in E$ and $f \in C(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*
2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$, $k_0 := \max\{f(x_1), f(x_2)\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$.*

Then $k_0 < k_1 < +\infty$.

Proof. [Rab86, page 7] Clearly $k_1 \in \mathbb{R}$, so we are left to prove that $k_1 > k_0$. Take $r \in \mathbb{R}^+$ as in Definition 5.11. Observe that, for each $\gamma \in \Gamma$,

$$\gamma([0,1]) \cap \{y \in E \ : \| y - x_1 \| = r\} \neq \emptyset$$

and therefore

$$\max_{t\in[0,1]} f(\gamma(t)) \geq \inf_{\|y-x_1\|=r} f(y).$$

Then

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) \geq \inf_{\|y-x_1\|=r} f(y) > \max\{f(x_1), f(x_2)\} := k_0.$$

■

Conditions *1.* and *2.* of Theorem 5.10 are not enough to imply that k_1 is a critical value of f as we can see with the following example.

Example 5.13 [GT01, page 36] The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2 + (x+1)^3 y^2$ satisfies the mountain pass geometry with $x_1 = (0,0)$, $x_2 = (-2,3)$ and $r = \frac{1}{2}$. $(0,0)$ is a strict local minimizer and is the only critical point of f. Therefore there is no $z \in \mathbb{R}^2$ such that $f(z) = k_1 > 0$ and $f'(z) = 0$.

In 1980 Brézis, Coron and Nirenberg obtained in [BCN80] a generalization of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for functionals satisfying the $(\mathbf{PS})_{k_1}$ condition:

Theorem 5.14 ***(Mountain Pass Theorem of Brézis-Coron-Nirenberg)*** *Let E be a real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*
2. $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ *and* $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;
3. *f satisfies $(\mathbf{PS})_{k_1}$.*

Then k_1 is a critical value of f.

We now present the following generalization of the Mountain Pass Theorem of Brézis-Coron-Nirenberg.

Theorem 5.15 ***(Mountain Pass Theorem with nonstandard Palais-Smale condition)*** [MNb] *Let E be a real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*
2. $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ *and* $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;
3. *f satisfies $(PS2)_{k_1}$.*

Then k_1 is a critical value of f.

The proof of this theorem can be easily obtained from the following consequence of Lemma 5.1 [Maw02, pages 17-18]:

Theorem 5.16 *Let E be a real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*
2. $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ *and* $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$.

Then, for each $\epsilon > 0$, $\delta > 0$ *and* $\gamma \in \Gamma$ *such that*

$$\max_{t\in[0,1]} f(\gamma(t)) \leq k_1 + \epsilon,$$

there exists $u \in E$ *such that*

(i) $k_1 - 2\epsilon \leq f(u) \leq k_1 + 2\epsilon$;

(ii) $dist(u, \gamma([0,1])) \leq 2\delta$;

(iii) $\| f'(u) \| < \frac{8\epsilon}{\delta}$.

Proof. By hypothesis there exists $r \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| > r$ and

$$k_0 := \max\{f(x_1), f(x_2)\} < \inf_{\|y-x_1\|=r} f(y) := b.$$

We can suppose, without loss of generality, that $\epsilon < \frac{b-k_0}{2}$. Then $k_0 < b - 2\epsilon \leq k_1 - 2\epsilon$. Fix $\delta > 0$ and choose $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} f(\gamma(t)) \leq k_1 + \epsilon$. Let $S := \gamma([0,1])$ and assume that

$$\forall y \in E \left[[\ k_1 - 2\epsilon \leq f(y) \leq k_1 + 2\epsilon \ \wedge \ dist(y, S) \leq 2\delta \ ] \ \Rightarrow \| f'(y) \| \geq \frac{8\epsilon}{\delta} \right].$$

Then, by Lemma 5.1, there exists $\eta \in C([0,1] \times E, E)$ such that

$$\eta(t, y) = y \quad \text{if} \quad y \notin f^{-1}([k_1 - 2\epsilon, k_1 + 2\epsilon]) \cap S_{2\delta}$$

and

$$\eta(1, f^{k_1+\epsilon} \cap S) \subseteq f^{k_1-\epsilon}.$$

Let $\beta(t) = \eta(1, \gamma(t))$ for all $t \in [0,1]$. Observe that β is continuous, $\beta(0) = \eta(1, x_1) = x_1$ and $\beta(1) = \eta(1, x_2) = x_2$ (because $f(x_i) \leq k_0 < k_1 - 2\epsilon$ for $i = 1, 2$). Hence, $\beta \in \Gamma$. Since $\max_{t\in[0,1]} f(\gamma(t)) \leq k_1 + \epsilon$, it follows that

$$\eta(1, \gamma([0,1])) \subseteq f^{k_1-\epsilon}$$

that is,

$$f(\beta([0,1])) = f(\eta(1, \gamma([0,1]))) \ \subseteq\ ]-\infty, k_1 - \epsilon]$$

which contradicts the definition of $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$. ∎

Corollary 5.17 *Let E be a real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*

2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$.*

Then, for each $\gamma \in {}^\star\Gamma$ such that

$$\max_{t\in^\star[0,1]} f(\gamma(t)) \approx k_1,$$

there exists $u \in {}^\star E$ such that

(i) $f(u) \approx k_1$;

(ii) $dist(u, \gamma({}^\star[0,1])) \approx 0$;

(iii) $f'(u) \approx 0$.

Proof. Apply the Transfer Principle to Theorem 5.16, take $0 < \epsilon \approx 0$ and $\delta = \sqrt{\epsilon}$. ■

Proof. (of Theorem 5.15) Let $\gamma_0 \in {}^\star\Gamma$ be such that

$$\max_{t\in^\star[0,1]} f(\gamma_0(t)) \approx k_1.$$

Then, by Corollary 5.17, there exists $u \in {}^\star E$ such that $f(u) \approx k_1$ and $f'(u) \approx 0$. Since f satisfies $(PS2)_{k_1}$, we may conclude that k_1 is a critical value of f. ■

To finish this section we will obtain the "dual" of Theorem 5.15.

Theorem 5.18 *Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *there exist $x_1, x_2 \in E$ and $r \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| > r$ and*

$$k_2 := \min\{f(x_1), f(x_2)\} > \sup_{\|y-x_1\|=r} f(y);$$

2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t))$;*

3. *f satisfies $(PS2)_{k_3}$.*

Then $k_3 < k_2$ and k_3 is a critical value of f.

Proof. By condition *1.* we obtain

$$-\min\{f(x_1), f(x_2)\} < -\sup_{\|y-x_1\|=r} f(y).$$

Hence

$$\max\{-f(x_1), -f(x_2)\} < \inf_{\|y-x_1\|=r} (-f(y))$$

and therefore $-f$ satisfies the mountain pass geometry with respect to x_1 and x_2. Since

$$-\sup_{\gamma\in\Gamma}\min_{t\in[0,1]} f(\gamma(t)) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} (-f(\gamma(t)))$$

it follows that

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} (-f(\gamma(t))) = -k_3.$$

By Proposition 5.12,

$$-k_3 > \max\{-f(x_1), -f(x_2)\}$$

then,

$$-k_3 > -\min\{f(x_1), f(x_2)\} = -k_2.$$

Since $-f$ satisfies $(PS2)_{-k_3}$, it follows by Theorem 5.15 that $-k_3$ is a critical value of $-f$. Therefore, $k_3 < k_2$ and k_3 is a critical value of f. ■

5.4 Obtaining almost critical points in real Hilbert spaces

In the following we will suppose that H is a real Hilbert space and U is an open subset of H.

Later on we will need the following lemmas. First define

$$r \gg s := r > s \quad \wedge \quad r \not\approx s \qquad (r, s \in {}^\star\mathbb{R})$$

and

$$C([0,1],[0,1]) := \{\delta : [0,1] \to [0,1] \ : \ \delta \text{ is continuous}\}$$

and recall that

$$ns({}^\star U) := \{x \in {}^\star H : \ x \in ns({}^\star H) \ \wedge \ st(x) \in U\}.$$

Lemma 5.19 [BMN+05] *Suppose H is a real Hilbert space with inner product $\cdot \bullet \cdot$ and norm $\| \cdot \|$ and let U be an open subset of H. Let $f \in C^1(U, \mathbb{R})$ and $x \in ns({}^\star U)$. If $f'(x) \not\approx 0$, then for every $0 < \varepsilon \approx 0$, the following inequality holds:*

$$f(x - \varepsilon f'(x)) < f(x) - \varepsilon \frac{\| f'(x) \|^2}{2}.$$

Proof. Suppose $x \in ns({}^\star U)$ is such that $f'(x) \not\approx 0$ and let $0 < \varepsilon \approx 0$. For some $\iota \approx 0$ in ${}^\star\mathbb{R}$,

$$\begin{aligned} f(x - \varepsilon f'(x)) &= f(x) - \varepsilon \left[f'(x) \bullet f'(x) - \iota \right] && (5.1) \\ &= f(x) - \varepsilon \left[\|f'(x)\|^2 - \iota \right] && (5.2) \\ &< f(x) - \varepsilon \frac{\| f'(x) \|^2}{2}. && (5.3) \end{aligned}$$

Condition (5.1) is true because f is of class C^1 and $x \in ns({}^\star H)$ (Theorem 2.37) and condition (5.3) results from

$$\| f'(x) \| \gg 0 \quad \wedge \quad \| f'(x) \|^2 - \iota \approx \| f'(x) \|^2 > \frac{\| f'(x) \|^2}{2} \quad \wedge \quad \varepsilon > 0$$

which proves the lemma. ∎

The following lemma can be used to get almost critical points for C^1 functionals defined in real Hilbert spaces.

Lemma 5.20 *Let H be a real Hilbert space with inner product $\cdot \bullet \cdot$ and norm $\| \cdot \|$. Suppose that $f \in C^1(H, \mathbb{R})$ satisfies the mountain pass geometry with respect to x_1 and x_2. Let*

$$\Gamma := \{ \gamma \in C([0,1], H) : \gamma(0) = x_1 \wedge \gamma(1) = x_2 \}$$

and

$$k_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)).$$

Then

$$\begin{aligned} \forall \gamma \in {}^\star\Gamma \Big[\big[\gamma({}^\star[0,1]) \subseteq ns({}^\star H) \ \wedge \ \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1 \big] \\ \Rightarrow \exists t_0 \in {}^\star[0,1] \ \big[f(\gamma(t_0)) \approx k_1 \ \wedge \ \| f'(\gamma(t_0)) \| \approx 0 \big] \Big]. \end{aligned}$$

Proof. [BMN+05] Take $\gamma \in {}^\star\Gamma$ such that

$$\gamma({}^\star[0,1]) \subseteq ns({}^\star H) \quad \wedge \quad k_2 := \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1 \tag{5.4}$$

and let $k_0 := \max\{f(x_1), f(x_2)\}$. Then

$$k_0 \quad < \quad k_1 \quad \leq \quad k_2 \quad \approx \quad k_1. \tag{5.5}$$

Define

$$U := \{t \in {}^\star[0,1] \ : \ k_1 \leq f(\gamma(t)) \leq k_2\} \tag{5.6}$$

and

$$d \ := \ \min\{\|f'(\gamma(t))\| \ : \ t \in U\}. \tag{5.7}$$

We claim that $d \approx 0$. If not define

$$V := \{t \in {}^\star[0,1] \ : \ \|f'(\gamma(t))\| > \frac{d}{2}\}$$

and

$$W := (\ {}^\star[0,1] \setminus V) \cup \{0,1\}.$$

Then W and U are *closed, V is *open (in the relative *topology of $^\star[0,1]$) and $U \subseteq V$. Moreover,

$$\{0,1\} \not\subseteq U$$

since $k_1 > \max\{f(\gamma(0)), f(\gamma(1))\}$, and

$$V \neq U$$

since $U \neq {}^\star[0,1]$ and $^\star[0,1]$ is *connected.

Hence, by Urysohn's Theorem and the Transfer Principle, there exists a function $u \in {}^\star C([0,1],[0,1])$ such that

$$u(W) = \{0\} \quad \text{and} \quad u(U) = \{1\}. \tag{5.8}$$

Choose b such that

$$0 \ \ \leq \tfrac{2(k_2 - k_1)}{d^2} \ \ < b \ \approx 0 \tag{5.9}$$

and define $\eta : {}^\star[0,1] \to [0,b]$ by

$$\eta(t) := bu(t);$$

let

$$\gamma_\eta(t) := \gamma(t) - \eta(t) f'(\gamma(t)).$$

Since $\gamma({}^\star[0,1]) \subseteq ns({}^\star H)$, $\eta(t) \approx 0$ for all $t \in {}^\star[0,1]$ and f' is continuous, it follows that $\gamma_\eta({}^\star[0,1]) \subseteq ns({}^\star H)$. Moreover, $\gamma_\eta \in {}^\star\Gamma$, because $\gamma_\eta \in {}^\star C([0,1], H)$, $\gamma_\eta(0) = x_1$ and $\gamma_\eta(1) = x_2$.

Next we will prove that under these conditions

$$\forall t \in {}^\star[0,1] \quad f(\gamma_\eta(t)) \ < \ k_1,$$

which will be a contradiction to the definition of k_1.

If $t \in W$, then

$$f(\gamma_\eta(t)) = f(\gamma(t)) < k_1,$$

because $\eta(t) = 0$ and $t \notin U$.

If $t \in U$, then

$$\begin{aligned} f(\gamma_\eta(t)) &= f(\gamma(t) - bf'(\gamma(t))) \\ &< f(\gamma(t)) - b\frac{\| f'(\gamma(t)) \|^2}{2} \qquad \text{(by Lemma 5.19)} \\ &\leq f(\gamma(t)) - b\frac{d^2}{2} \qquad \text{(by (5.7))} \\ &< f(\gamma(t)) - (k_2 - k_1) \qquad \text{(by (5.9))} \\ &\leq k_1 \qquad \text{(by (5.6))}. \end{aligned}$$

Finally, if $t \in V \setminus U$, Lemma 5.19 and the definition of U imply

$$f(\gamma_\eta(t)) \leq f(\gamma(t)) - \eta(t)\frac{\| f'(\gamma(t)) \|^2}{2} < k_1$$

which, as pointed out above, is a contradiction and we conclude that d must be infinitesimal. Hence, there exists $t_0 \in U$ such that $\| f'(\gamma(t_0)) \| \approx 0$, that is,

$$\exists t_0 \in {}^\star[0,1] \ [\ f(\gamma(t_0)) \approx k_1 \wedge \ \| f'(\gamma(t_0)) \| \approx 0 \ ],$$

proving the desired result. ■

Lemma 5.20 was inspired in a claim that appears in [JLJ98, pages 64-66] about coercive functionals defined in a finite real Banach space and where x_1 and x_2 are two strict local minimizers. As an historical remark, we mention that Courant [Cou50] proved in 1950 the following theorem:

Suppose that $f \in C^1(\mathbb{R}^n, \mathbb{R})$ is coercive and possesses two distinct strict local minimizers x_1 and x_2. Then f possesses a third critical point x_3 distinct from x_1 and x_2, characterized by

$$f(x_3) = \inf_{K \in \Sigma} \max_{x \in K} f(x),$$

where

$$\Sigma = \{K \subseteq \mathbb{R}^n \ : \ K \text{ is compact and connected and } x_1, x_2 \in K\}.$$

Moreover, x_3 is not a local minimizer.

Notice that, since x_1 and x_2 are two strict local minimizers, then f satisfies the mountain pass geometry with respect to x_1 and x_2. Also note that,

- $\inf_{K \in \Sigma} \max_{x \in K} f(x) \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$, since $\Gamma \subseteq \Sigma$;
- $\inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)) \leq \inf_{K \in \Sigma} \max_{x \in K} f(x)$, because for each $K \in \Sigma$ and $\epsilon \in \mathbb{R}^+$, there exists $\gamma \in \Gamma$ such that K is a uniform ϵ-neighborhood of $\gamma([0,1])$.

Hence,

$$\inf_{K \in \Sigma} \max_{x \in K} f(x) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$$

and this result of Courant may be considered as an *ancestor* of the Mountain Pass Theorem of Ambrosetti-Rabinowitz.

Remark 5.21 We notice that Lemma 5.20 can also be proved using the Quantitative Deformation Lemma, more precisely, Corollary 5.17, because for any $\gamma \in {}^\star\Gamma$ such that

$$\gamma({}^\star[0,1]) \subseteq ns({}^\star H) \quad \wedge \quad \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1,$$

there exists, by Corollary 5.17, $u \in {}^\star H$ such that

$$f(u) \approx k_1 \quad \wedge \quad dist(u, \gamma({}^\star[0,1])) \approx 0 \quad \wedge \quad f'(u) \approx 0.$$

Therefore, there exists $t_0 \in {}^\star[0,1]$ such that $u \approx \gamma(t_0) \in ns({}^\star H)$. Since f and f' are continuous,

$$f(\gamma(t_0)) \approx k_1 \quad \wedge \quad \| f'(\gamma(t_0)) \| \approx 0$$

and the lemma is proved.

5.5 A Mountain Pass Theorem in the finite dimensional case

In this section we will use Lemma 5.20 to give a nonstandard proof of the Mountain Pass Theorem of Ambrosetti-Rabinowitz for the special case where the functional is coercive and is defined in a finite dimensional real Banach space.

Until the end of this chapter we will suppose that E is a finite dimensional real Banach space.

In Chapter 4 we proved that

> *If E is a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$ is coercive, then f satisfies $(PS1)$.*

As $(PS1) \Rightarrow$ (**PS**) (Theorem 4.6), the following may be derived as a consequence of the Mountain Pass Theorem of Ambrosetti-Rabinowitz. Nevertheless, we will give a proof which does not use Palais-Smale conditions but only some properties of the coercive functionals.

Theorem 5.22 ***(Mountain Pass Theorem - a special case)*** *Let E be a finite dimensional real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*
2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;*
3. *f is coercive.*

Then k_1 is a critical value of f.

Proof. Take $\gamma_0 \in {}^\star\Gamma$ such that $\max_{t\in^\star[0,1]} f(\gamma_0(t)) \approx k_1$. First of all note that, as f is coercive,

$$\gamma_0({}^\star[0,1]) \subseteq fin({}^\star E)$$

and since E is finite dimensional,

$$\gamma_0({}^\star[0,1]) \subseteq ns({}^\star E).$$

Then, by Lemma 5.20,

$$\exists t_0 \in {}^{\star}[0,1] \; [\; f(\gamma_0(t_0)) \approx k_1 \quad \wedge \quad \| \; f'(\gamma_0(t_0)) \; \| \approx 0 \;].$$

The continuity of f and f' shows that

$$f(st(\gamma_0(t_0))) = st(f(\gamma_0(t_0))) = k_1 \quad \wedge \quad f'(st(\gamma_0(t_0))) = st(f'(\gamma_0(t_0))) = 0;$$

hence k_1 is a critical value of f. ■

In Appendix B we present other nonstandard proofs for this theorem.

Remark 5.23 If the coercivity of the functional is removed, we cannot guarantee that k_1 is a critical value of f, as can be seen with the function $f(x,y) = x^2 + (x+1)^3 y^2$ of Example 5.13; f is not coercive since for all $y \approx +\infty$, $\| \; (-2,y) \; \| \approx +\infty$ and

$$f(-2,y) = 4 - y^2 \approx -\infty.$$

The following example shows that the coercivity of the functional is not a necessary condition for k_1 to be a critical value of f.

Example 5.24 Let $h(x,y) = \sin(x^2+y^2)$ for all $(x,y) \in \mathbb{R}^2$. Clearly h is C^1, satisfies the mountain pass geometry with respect to $(0,0)$, $(0,\sqrt{\pi})$ and $r = \sqrt{\frac{\pi}{2}}$, h is not coercive, but $k_1 = 1$ is a critical value of h.

5.6 Mountain Pass Theorems without Palais-Smale conditions

In this section we will prove two mountain pass type results for functionals that satisfy special properties. Notice that the following theorems cannot be obtained from the Mountain Pass Theorem of Ambrosetti-Rabinowitz or generalizations thereof, e.g. given in [GT01], since conditions *3.* of Theorem 5.25 and Theorem 5.29 do not imply (**PS**) or weaker forms of (**PS**).

Theorem 5.25 ***(Mountain Pass Theorem without Palais-Smale conditions in finite dimension)***[Mar05] *Let E be a finite dimensional real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E,\mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*

2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;*

3. *there exists $s \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| < s$ and if $\| x - x_1 \| \geq s$ then $f(x) < k_1$.*

Then k_1 is a critical value of f.

Proof. Let $\gamma_0 \in {}^\star\Gamma$ be such that $k_1 \leq \max_{t\in^\star[0,1]} f(\gamma_0(t)) \approx k_1$. Using condition *3.* we may assume that $\gamma_0(^\star[0,1]) \subseteq \overline{\mathbf{B}}_s(x_1)$ and, since E is finite dimensional, $\gamma_0(^\star[0,1]) \subseteq ns(^\star E)$. Therefore, by Lemma 5.20, there exists $t_0 \in^\star[0,1]$ such that

$$f(\gamma_0(t_0)) \approx k_1 \quad \wedge \quad \| f'(\gamma_0(t_0)) \| \approx 0.$$

The continuity of f and f' shows that $st(\gamma_0(t_0))$ is a critical point with value k_1. ■

We believe that our Theorem 5.25 is sometimes easier to apply than the Mountain Pass Theorem of Ambrosetti-Rabinowitz in the n-dimensional case: the third condition of Theorem 5.25 may be simpler to verify than (**PS**), even for small n. Moreover, there are functionals which satisfy all the conditions of Theorem 5.25 but do not satisfy (**PS**).

Example 5.26 The function $g(x,y) = (x^2+y^2+\sin(x^2+y^2))(1-(x^2+y^2))$ for all $(x,y) \in \mathbb{R}^2$ is C^1 and verifies the mountain pass geometry with respect to $z_1 = (0,0)$ and $z_2 = (1,0)$ since

$$\inf_{\|(x,y)\|=\frac{1}{2}} g(x,y) > k_0 := \max\{g(z_1), g(z_2)\} = 0.$$

Since g is not coercive we cannot apply Theorem 5.22. But is easy to prove that

$$\| (x,y) \| \geq 4 \Rightarrow g(x,y) < 0 < k_1$$

and therefore, by Theorem 5.25, we can conclude that k_1 is a critical value of g. We also point out that checking the (**PS**) condition may not be an easy task.

Next we present an example of a functional defined in $\mathbb{R}^2$ that satisfies all the conditions of Theorem 5.25 but does not satisfy (**PS**).

Example 5.27 Let $h(x,y) = [1-(x^2+y^2)]\exp^{-(x^2+y^2)}\arctan(x^2+y^2)$ for all $(x,y)\in\mathbb{R}^2$. Clearly h is a C^1 functional, $h(0,0)=0$, $h(1,0)=0$,

$$\inf_{\|(x,y)\|=\frac{1}{2}} h(x,y) > k_0 := \max\{h(0,0), h(1,0)\} = 0$$

and

$$(x,y)\notin \mathbf{B}_2(0,0) \Rightarrow h(x,y)<0.$$

The function h does not satisfy (**PS**) condition, since the sequence $((n,n))_{n\in\mathbb{N}}$ is such that $(h(n,n))_{n\in\mathbb{N}}$ is bounded, $\frac{\partial h}{\partial x}(n,n)\to 0$ and $\frac{\partial h}{\partial y}(n,n)\to 0$ but the sequence $((n,n))_{n\in\mathbb{N}}$ does not contain a convergent subsequence.

We proceed presenting the "dual" of Theorem 5.25.

Theorem 5.28 [Mar05] *Let E be a finite dimensional real Banach space and $f\in C^1(E,\mathbb{R})$. Suppose that*

1. *there exist $x_1,x_2\in E$ and $r\in\mathbb{R}^+$ such that $\| x_2-x_1 \| > r$ and*

$$k_2 := \min\{f(x_1), f(x_2)\} > \max_{\|y-x_1\|=r} f(y);$$

2. *$\Gamma := \{\gamma\in C([0,1],E) : \gamma(0)=x_1 \wedge \gamma(1)=x_2\}$ and $k_3 := \sup_{\gamma\in\Gamma}\min_{t\in[0,1]} f(\gamma(t))$;*

3. *there exists $s\in\mathbb{R}^+$ such that $\| x_2-x_1 \| < s$ and if $\| x-x_1 \| \geq s$ then $f(x)>k_3$.*

Then $k_3<k_2$ and k_3 is a critical value of f.

Proof. Apply Theorem 5.25 to $-f$ and use the same arguments used in the proof of Theorem 5.18. ■

The following theorem is an easy consequence of Lemma 5.20. Note also that Theorem 5.22 follows easily from Theorem 5.29.

Theorem 5.29 ***(Mountain Pass Theorem without Palais-Smale conditions in Hilbert spaces)*** *Let H be a real Hilbert space, $x_1,x_2\in H$ and $f\in C^1(H,\mathbb{R})$. Suppose that*

1. *f satisfies the mountain pass geometry with respect to x_1 and x_2;*

2. $\Gamma := \{\gamma \in C([0,1],H) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ *and* $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;

3. $\exists \gamma \in {}^\star\Gamma$ $[\ \gamma({}^\star[0,1]) \subseteq ns({}^\star H)\ \wedge\ \max_{t\in^\star[0,1]} f(\gamma(t)) \approx k_1\ ]$.

Then k_1 is a critical value of f.

The following example shows that Theorem 5.29 cannot be obtained from the Mountain Pass Theorem of Ambrosetti-Rabinowitz.

Example 5.30 Consider the function h of Example 5.27:

$$h(x,y) = [1-(x^2+y^2)] \exp^{-(x^2+y^2)} \arctan(x^2+y^2) \qquad ((x,y) \in \mathbb{R}^2).$$

We saw that this function satisfies the mountain pass geometry with respect to $(0,0)$ and $(1,0)$ and

$$(x,y) \notin \mathbf{B}_2(0,0) \Rightarrow h(x,y) < 0.$$

Since $k_1 > 0$, there exists $\gamma \in {}^\star\Gamma$ such that

$$\max_{t\in^\star[0,1]} h(\gamma(t)) \approx k_1 \quad \wedge \quad \gamma({}^\star[0,1]) \subseteq \mathbf{B}_2(0,0) \subseteq ns({}^\star\mathbb{R}^2).$$

Hence, h satisfies all the conditions of Theorem 5.29 but do not satisfy (**PS**) condition.

To finish this chapter we present the "dual" of Theorem 5.29. Since the proof is obvious we omit it.

Theorem 5.31 *Let H be a real Hilbert space and $f \in C^1(H,\mathbb{R})$. Suppose that*

1. *there exist $x_1, x_2 \in H$ and $r \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| > r$ and*

$$k_2 := \min\{f(x_1), f(x_2)\} > \sup_{\|y-x_1\|=r} f(y);$$

2. $\Gamma := \{\gamma \in C([0,1],H) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ *and* $k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t))$;

3. $\exists \gamma \in {}^\star\Gamma$ $[\ \gamma({}^\star[0,1]) \subseteq ns({}^\star H)\ \wedge\ \min_{t\in^\star[0,1]} f(\gamma(t)) \approx k_3\ ]$.

Then k_3 is a critical value of f.

Chapter 6

Three Critical Points Theorems

6.1 Introduction

The goal of this chapter is to present new variants of Three Critical Points Theorems. Namely, we will prove

- Three Critical Points Theorems with $(PS2)_c$ condition (Theorem 6.1, Theorem 6.2, Theorem 6.4 and Theorem 6.5);
- Three Critical Points Theorems without Palais-Smale conditions in finite dimensional spaces (Theorem 6.7, Theorem 6.8, Theorem 6.9 and Theorem 6.10);
- Three Critical Points Theorems without Palais-Smale conditions in real Hilbert spaces (Theorem 6.12, Theorem 6.13, Theorem 6.14 and Theorem 6.15).

Let us recall some notation used in the last chapter. If E is a real Banach space, $f : E \to \mathbb{R}$ a continuous functional and x_1 and x_2 are two elements of E, we will denote

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \ \wedge \ \gamma(1) = x_2\}$$

$$k_0 := \max\{f(x_1), f(x_2)\}$$

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

$$k_2 := \min\{f(x_1), f(x_2)\}$$

and

$$k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t)).$$

6.2 Three Critical Points Theorems with a nonstandard Palais-Smale condition

The Three Critical Points Theorems presented in this section are obtained from the Mountain Pass Theorem with nonstandard Palais-Smale condition (Theorem 5.15) and Theorem 5.18.

Theorem 6.1 *Let E be a real Banach space and $f \in C^1(E,\mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local minimizers of f,*

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and f satisfies $(PS2)_{k_1}$. Then f has a third critical point with value k_1.

Proof. Since x_1 and x_2 are two strict local minimizers, f satisfies the mountain pass geometry with respect to x_1 and x_2. Theorem 5.15 shows that k_1 is a critical value of f; since

$$k_1 > k_0 := \max\{f(x_1), f(x_2)\}$$

there exists another critical point x_3 such that $f(x_3) = k_1$. ■

Theorem 6.1 can be generalized in the following sense.

Theorem 6.2 *Let E be a real Banach space and $f \in C^1(E,\mathbb{R})$. Suppose x_1 and x_2 are two distinct local minimizers of f,*

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and f satisfies $(PS2)_{k_1}$. Then f has (at least) one critical point with value k_1 or f has an infinite number of critical points with value $k_0 := \max\{f(x_1), f(x_2)\}$.

Proof. Suppose that $f(x_2) \le f(x_1) := k_0$.

If x_1 is a strict local minimizer then, by Theorem 5.15, there exists a third critical point x_3 such that

$$f(x_3) = k_1 > k_0 := \max\{f(x_1), f(x_2)\}.$$

If x_1 is a non strict local minimizer then, in any neighborhood of x_1, there exists $y \neq x_1$ such that $f(y) = f(x_1) = k_0$; hence, y is also a local minimizer of f. Since in any real Banach space every local minimizer (or maximizer) of a Fréchet differentiable functional is a critical point, we may conclude that f has an infinite number of critical points with value k_0. ■

Next we present a functional that satisfies all the conditions of Theorem 6.2 and for which the number of critical points with value k_0 is (only) countable.

Example 6.3 Consider the function

$$f(x) = \begin{cases} x^4 \sin^2(\frac{1}{x}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

Observe that $f \geq 0$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = f(\frac{1}{\pi}) = 0 := k_0$ and f is coercive (hence, f satisfies $(PS2)$). Also note that

1. $f(x) = 0 \ \Leftrightarrow \ x \in \left\{\frac{1}{k\pi} \ : \ k \in \mathbb{Z} \setminus \{0\}\right\} \cup \{0\}$;
2. $\frac{1}{\pi}$ is a strict minimizer;
3. 0 is a non strict minimizer.

Therefore, there exists only a countable number of critical points with value k_0.

We finish this section with some consequences of Theorem 5.18.

Theorem 6.4 *Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local maximizers of f,*

$$k_3 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} f(\gamma(t))$$

and f satisfies $(PS2)_{k_3}$. Then f has a third critical point with value k_3.

Proof. Since f satisfies all the conditions of Theorem 5.18, we may conclude that k_3 is a critical value of f. Since

$$k_3 < k_2 := \min\{f(x_1), f(x_2)\},$$

there exists another critical point x_3 such that $f(x_3) = k_3$. ■

Theorem 6.5 *Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct local maximizers of f,*

$$k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t))$$

and f satisfies $(PS2)_{k_3}$. Then f has (at least) one critical point with value k_3 or f has an infinite number of critical points with value $k_2 := \min\{f(x_1), f(x_2)\}$.

Proof. Apply Theorem 5.18 and use the same arguments used in the proof of Theorem 6.2. ■

Example 6.6 Let f be the function of Example 6.3, $x_1 = 0$ and $x_2 = \frac{1}{\pi}$. Thus, $-f$ shows that, in the conditions of Theorem 6.5, the number of critical points with value $k_2 = 0$ may be only countable.

6.3 Three Critical Points Theorems without Palais-Smale conditions

The Three Critical Points Theorems proved in this section are obtained from the Mountain Pass Theorems without Palais-Smale conditions (Theorem 5.25 and Theorem 5.29) and their "duals" (Theorem 5.28 and Theorem 5.31). The proofs are similar to the proofs done in the last section so we will omit them.

Theorem 6.7 [Mar05] *Let E be a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local minimizers of f,*

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists s \in \mathbb{R}^+ \ [\ \| x_2 - x_1 \| < s \ \wedge \ \forall x \in E \ (\ \| x - x_1 \| \geq s \Rightarrow f(x) < k_1 \ )\ ].$$

Then f has a third critical point with value k_1.

Theorem 6.8 [Mar05] *Let E be a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct local minimizers of f,*

$$k_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists s \in \mathbb{R}^+ \ [\ \| x_2 - x_1 \| < s \ \wedge \ \forall x \in E \ (\ \| x - x_1 \| \geq s \Rightarrow f(x) < k_1 \ )\ ].$$

Then f has (at least) one critical point with value k_1 or f has an infinite number of critical points with value $k_0 := \max\{f(x_1), f(x_2)\}$.

Theorem 6.9 [Mar05] *Let E be a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local maximizers of f,*

$$k_3 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists s \in \mathbb{R}^+ \ [\ \| x_2 - x_1 \| < s \ \wedge \ \forall x \in E \ (\ \| x - x_1 \| \geq s \Rightarrow f(x) > k_3 \ )\ ].$$

Then f has a third critical point with value k_3.

Theorem 6.10 [Mar05] *Let E be a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose x_1 and x_2 are two distinct local maximizers of f,*

$$k_3 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists s \in \mathbb{R}^+ \ [\ \| x_2 - x_1 \| < s \ \wedge \ \forall x \in E \ (\ \| x - x_1 \| \geq s \Rightarrow f(x) > k_3 \ )\ ].$$

Then f has (at least) one critical point with value k_3 or f has an infinite number of critical points with value $k_2 := \min\{f(x_1), f(x_2)\}$.

Remark 6.11 It is not difficult to see that there exist functionals that satisfy all the conditions of Theorem 6.8 (respectively, Theorem 6.10) and such that the number of critical points with value k_0 (respectively, k_2) is only countable.

Theorem 6.12 *Let H be a real Hilbert space and $f \in C^1(H, \mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local minimizers of f,*

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists \gamma \in {}^\star\Gamma \; [\; \gamma({}^\star[0,1]) \subseteq ns({}^\star H) \;\wedge\; \max_{t\in^\star[0,1]} f(\gamma(t)) \approx k_1 \;].$$

Then f has a third critical point with value k_1.

Theorem 6.13 *Let H be a real Hilbert space and $f \in C^1(H, \mathbb{R})$. Suppose x_1 and x_2 are two distinct local minimizers of f,*

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists \gamma \in {}^\star\Gamma \; [\; \gamma({}^\star[0,1]) \subseteq ns({}^\star H) \;\wedge\; \max_{t\in^\star[0,1]} f(\gamma(t)) \approx k_1 \;].$$

Then f has (at least) one critical point with value k_1 or f has an infinite number of critical points with value $k_0 := \max\{f(x_1), f(x_2)\}$.

Theorem 6.14 *Let H be a real Hilbert space and $f \in C^1(H, \mathbb{R})$. Suppose x_1 and x_2 are two distinct strict local maximizers of f,*

$$k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists \gamma \in {}^\star\Gamma \; [\; \gamma({}^\star[0,1]) \subseteq ns({}^\star H) \;\wedge\; \min_{t\in^\star[0,1]} f(\gamma(t)) \approx k_3 \;].$$

Then f has a third critical point with value k_3.

Theorem 6.15 *Let H be a real Hilbert space and $f \in C^1(H,\mathbb{R})$. Suppose x_1 and x_2 are two distinct local maximizers of f,*

$$k_3 := \sup_{\gamma\in\Gamma} \min_{t\in[0,1]} f(\gamma(t))$$

and f satisfies the condition

$$\exists\gamma \in {}^{\star}\Gamma\ [\ \gamma({}^{\star}[0,1]) \subseteq ns({}^{\star}H)\ \wedge \min_{t\in{}^{\star}[0,1]} f(\gamma(t)) \approx k_3\ ].$$

Then f has (at least) one critical point with value k_3 or f has an infinite number of critical points with value $k_2 := \min\{f(x_1), f(x_2)\}$.

The observations in Remark 6.11 are equally applied to Theorem 6.13 and Theorem 6.15.

Appendix A

Peano's Existence Theorem

Theorem A.1 ***(Nonstandard Peano's Existence Theorem)*** [MNa] *Suppose $F : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}$ is internal, S-bounded and $\alpha \in {}^\star\mathbb{R}$. Then there exists one and only one internal S-absolutely continuous function $X : \mathbb{T} \to {}^\star\mathbb{R}$ such that*

$$\begin{cases} X'(t) &= F(t, X(t)) \quad (t \in \mathbb{T} \setminus \{1 - \triangle\}) \\ X(0) &= \alpha \end{cases} . \tag{A.1}$$

Moreover, if α is finite, then $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_{fin}$.

Proof. Define $X : \mathbb{T} \to {}^\star\mathbb{R}$ recursively by

$$\begin{aligned} X(0) &= \alpha \\ X(t + \triangle) &= X(t) + F(t, X(t))\triangle \qquad (t \in \mathbb{T} \setminus \{1 - \triangle\}). \end{aligned}$$

X is internal, by construction

$$X(t) = \alpha + \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle \quad (t = k\triangle \in \mathbb{T}) \tag{A.2}$$

and

$$X'(t) = \frac{X(t + \triangle) - X(t)}{\triangle} = F(t, X(t)) \quad (t \in \mathbb{T} \setminus \{1 - \triangle\}).$$

We will proof that X is S-Lipschitzian. Take $M \in \mathbb{R}$ such that $|F(t,z)| \leq M$ for all $(t,z) \in \mathbb{T} \times {}^\star\mathbb{R}$. Suppose that $s = k_1\triangle < k_2\triangle = t$, for certain $k_1, k_2 \in \{0, 1, \cdots, N-1\}$. Then

$$\begin{aligned} |X(t) - X(s)| &= |\sum_{i=k_1}^{k_2-1} F(i\triangle, X(i\triangle))\triangle| \\ &\leq \sum_{i=k_1}^{k_2-1} |F(i\triangle, X(i\triangle))|\triangle \\ &\leq \sum_{i=k_1}^{k_2-1} M\triangle \\ &= M(t-s). \end{aligned}$$

Hence,

$$|X(t) - X(s)| \leq M|t-s| \qquad (s, t \in \mathbb{T}) \tag{A.3}$$

and therefore X is S-absolutely continuous (see Proposition 2.34).

The uniqueness of the internal function $X : \mathbb{T} \to {}^\star\mathbb{R}$ such that (A.1) holds is obvious.

Using (A.2) we can prove that, for each $k = 0, 1, \cdots, N-1$

$$|X(k\triangle) - \alpha| \leq \sum_{i=0}^{k-1} |F(i\triangle, X(i\triangle))|\triangle \leq \sum_{i=0}^{k-1} M\triangle \leq M$$

hence, $X(\mathbb{T}) \subseteq [\alpha - M, \alpha + M]$. If α is finite, we conclude that $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_{fin}$. ∎

Next we will present a nonstandard proof of Peano's Existence Theorem (a classical proof of this theorem can be found in [CL55], pages 6-7).

Theorem A.2 ***(Peano's Existence Theorem)*** *Suppose $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous and $x_0 \in \mathbb{R}$. Then there exists $x : [0,1] \to \mathbb{R}$ such that*

$$\begin{cases} x'(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{cases} . \tag{A.4}$$

Proof. Suppose $F = {}^\star f_{|_{\mathbb{T} \times {}^\star\mathbb{R}}} : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}$. F is internal, $F(\mathbb{T} \times {}^\star\mathbb{R}) \subseteq {}^\star\mathbb{R}_{fin}$ and for each $\tau \in \mathbb{T}$ and $y \in {}^\star\mathbb{R}_{fin}$,

$${}^\circ F(\tau, y) = f({}^\circ\tau, {}^\circ y);$$

note that

$$F(\tau, y) = {}^\star f(\tau, y) \approx f({}^\circ\tau, {}^\circ y)$$

because $\tau \approx {}^\circ\tau \in [0,1]$, $y \approx {}^\circ y \in \mathbb{R}$ and f is continuous.

By Nonstandard Peano's Existence Theorem (Theorem A.1), there exists an internal S-absolutely continuous function

$$X : \mathbb{T} \to {}^\star\mathbb{R}_{fin}$$

such that

$$\begin{cases} X'(\tau) &= F(\tau, X(\tau)) \qquad (\tau \in \mathbb{T} \setminus \{1 - \triangle\}) \\ X(0) &= x_0 \end{cases}.$$

Define $x : [0,1] \to \mathbb{R}$ such that

$$x({}^\circ\tau) = {}^\circ X(\tau) \qquad (\tau \in \mathbb{T}).$$

By Theorem 2.35, x is absolutely continuous. Moreover,

$$x(0) = {}^\circ X(0) = x_0,$$

so that x satisfies the initial condition.

Using the definition and continuity of x we have that

$$X(\tau) \approx x({}^\circ\tau) \approx x(\tau) \qquad (\tau \in \mathbb{T})$$

and, since f is continuous,

$$f(\tau, X(\tau)) \approx f(\tau, x(\tau)) \approx f({}^\circ\tau, x({}^\circ\tau)) \qquad (\tau \in \mathbb{T}).$$

Hence $G : \mathbb{T} \to {}^\star\mathbb{R}$ such that $G(\tau) = f(\tau, X(\tau))$ is a lifting of the Lebesgue integrable function $g : [0,1] \to \mathbb{R}$, $g(t) = f(t, x(t))$.

Moreover, G is S-integrable since, for all $A \in \mathcal{A}$,

$$\int_A \mid G \mid d\nu \leq \int_A M d\nu = M\nu(A)$$

where $M \in \mathbb{R}$ is an upper bound of f, and then

$$\int_{\mathbb{T}} \mid G \mid d\nu \leq M$$

and

$$\int_A | G | d\nu \approx 0$$

whenever $\nu(A) \approx 0$.

Next, we will prove that x is a solution to the initial value problem (A.4).

Fix $z \in [0,1]$ and $\tau = k\triangle \in \mathbb{T}$ such that $\tau \approx z$. Since

$$\begin{aligned} x(z) &= {}^\circ X(\tau) \\ &= x_0 + {}^\circ\left(\textstyle\sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle\right) \\ &= x_0 + {}^\circ\left(\textstyle\sum_{i=0}^{k-1} G(i\triangle)\triangle\right) \\ &= x_0 + \textstyle\int_{[0,z]} f(t, x(t)) d\lambda(t) \qquad \text{(Theorem 2.66)} \end{aligned}$$

then, $x(0) = x_0$ and $x'(t) = f(t, x(t))$ for all $t \in [0,1]$. ■

The reader can find an alternative nonstandard proof of Peano's Existence Theorem in [Cut00, pages 25-26]; in this proof, instead of using the nonstandard discrete derivative, the author uses an *infinitesimal delayed equation*.

Appendix B

Nonstandard proofs of a Mountain Pass Theorem in finite dimension

In this appendix we will present (other) nonstandard proofs of Theorem 5.22:

Theorem B.1 ***(Mountain Pass Theorem - a special case)*** *Let E be a finite dimensional real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose that*

1. *there exist $x_1, x_2 \in E$ and $r \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| > r$ and*

$$k_0 := \max\{f(x_1), f(x_2)\} \; < \; \min_{\|y - x_1\| = r} f(y);$$

2. *$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$;*

3. *f is coercive.*

Then k_1 is a critical value of f.

B.0.1 A "normal" proof

We begin this section proving the following lemma:

Lemma B.2 [BMN+05] *Suppose H is a real Hilbert space with inner product $\cdot \bullet \cdot$ and norm $\|\cdot\|$. Let $\gamma \in C([0,1], H)$, $r \in \mathbb{R}^+$ and $f \in C^1(H, \mathbb{R})$ such that*

$$f \circ \gamma \quad \textit{is not constant} \qquad \wedge \qquad 0 \ < \ r \ < \ \max_{t\in[0,1]} \|f'(\gamma(t))\|.$$

For each function $\eta : [0,1] \to \mathbb{R}_0^+$, define

$$\gamma_\eta(t) \ := \ \gamma(t) - \eta(t) f'(\gamma(t))$$

and

$$V_r := \ \{t \in \,]0,1[\ : \| \ f'(\gamma(t)) \ \| > r\}.$$

There exists a function $\delta_r \in C([0,1],[0,1])$ such that

$$\delta_r(0) \ = \delta_r(1) = \ 0 \tag{B.1}$$

$$\forall t \in V_r \quad \delta_r(t) > 0 \tag{B.2}$$

$$\forall t \in V_r \quad f(\gamma_{\delta_r}(t)) \ < \ f(\gamma(t)) \tag{B.3}$$

and for all functions $\eta : [0,1] \to \mathbb{R}_0^+$,

$$\forall t \in [0,1] \ \ [\ \eta(t) \le \delta_r(t) \quad \Rightarrow \quad f(\gamma_\eta(t)) \ \le f(\gamma(t)) \ ] \, . \tag{B.4}$$

Proof. In the first place we show that there exists $\varepsilon_r \in \,]0,1]$ such that for all functions $\eta : [0,1] \to \mathbb{R}^+$,

$$\forall t \in V_r \quad [\ \eta(t) \le \varepsilon_r \quad \Rightarrow \ f(\gamma_\eta(t)) < f(\gamma(t)) \ ] \, . \tag{B.5}$$

Note that, since $\gamma([0,1])$ is compact, $\gamma({}^\star[0,1]) \subseteq ns({}^\star H)$ (Theorem 2.22), so that condition (B.5) is verified for any positive infinitesimal ε_r by Lemma 5.19; by Overflow Principle (Theorem 2.16), condition (B.5) is indeed satisfied by a real $0 < \varepsilon_r < 1$; take such an ε_r, pick $\nu \in C([0,1],[0,1])$ such that

$$\nu(t) = \begin{cases} 0 & \text{if} \quad t \in [0,1] \setminus V_r \\ \in \,]0,1] & \text{if} \quad t \in V_r \end{cases}$$

and define

$$\delta_r(t) \ := \ \nu(t)\varepsilon_r \quad (t \in [0,1]).$$

Therefore, $\delta_r : [0,1] \to [0,1]$ is continuous, $\delta_r(0) = \nu(0)\varepsilon_r = 0$, $\delta_r(1) = \nu(1)\varepsilon_r = 0$ and, for each $t \in V_r$, $\delta_r(t) > 0$. Since, for each $t \in V_r$, $\delta_r(t) = \nu(t)\varepsilon_r \le \varepsilon_r$ then, by (B.5),

$$f(\gamma_{\delta_r}(t)) = f(\gamma(t) - \delta_r f'(\gamma(t))) < f(\gamma(t))$$

proving (B.3).

Let $\eta : [0,1] \to \mathbb{R}_0^+$ and $t \in [0,1]$ be such that $\eta(t) \leq \delta_r(t)$. If $\eta(t) = 0$, it is obvious that $f(\gamma_\eta(t)) = f(\gamma(t))$. Observe that, if $t \notin V_r$, $\delta_r(t) = 0$ and, consequently, $\eta(t) = 0$. Suppose then $\eta(t) \neq 0$ and $t \in V_r$. Since $\eta(t) \leq \delta_r(t) \leq \varepsilon_r$ then, by (B.5), $f(\gamma_\eta(t)) < f(\gamma(t))$, proving (B.4). ∎

Proof. (of Theorem B.1) [BMN+05] Take $\gamma \in {}^\star\Gamma$ such that $k_2 := \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1$. Let U, V, W and d as defined in the proof of Lemma 5.20, that is,

$$U := \{t \in {}^\star[0,1] \ : \ k_1 \leq f(\gamma(t)) \leq k_2\},$$

$$d \ := \ \min\{\|f'(\gamma(t))\| \ : \ t \in U\},$$
$$V := \{t \in {}^\star[0,1] \ : \ \|f'(\gamma(t))\| > \frac{d}{2}\}$$

and

$$W := (\ {}^\star[0,1] \setminus V) \cup \{0,1\}.$$

Suppose $d \not\approx 0$. Fix b such that

$$0 \leq \frac{2(k_2 - k_1)}{d^2} < b \ \approx 0$$

and let $\delta_{\frac{d}{2}} \in {}^\star C([0,1],[0,1])$ be as in Lemma B.2 with $r = \frac{d}{2}$. Note that, by (B.1) and (B.2),

$$\delta_{\frac{d}{2}}(0) \ = \delta_{\frac{d}{2}}(1) = 0 \quad \text{ and } \quad \forall t \in U \ \delta_{\frac{d}{2}}(t) > 0.$$

Define

$$\begin{aligned} \xi : {}^\star[0,1] \ &\to \ [0,b] \\ t \ &\mapsto \ \xi(t) := b\delta_{\frac{d}{2}}(t) \end{aligned}$$

and observe that

$$\forall t \in {}^\star[0,1] \quad \Big[0 \ \leq \ \xi(t) \leq \min\{\delta_{\frac{d}{2}}(t), \ b\} \quad \wedge \quad \xi(t) \approx 0\Big];$$

so that, by (B.4) and Transfer Principle,

$$\forall t \in {}^\star[0,1] \quad f(\gamma_\xi(t)) \ \leq \ f(\gamma(t)). \tag{B.6}$$

Let $u \in {}^\star C([0,1],[0,1])$ be as in (5.8), that is,

$$u(W) = \{0\} \quad \text{and} \quad u(U) = \{1\}.$$

Define

$$\begin{aligned}\delta(t) &:= \max\{bu(t),\ \xi(t)\}\\ &= b\cdot\max\{u(t),\ \delta_{\frac{d}{2}}(t)\}\end{aligned}$$

and

$$\gamma_\delta(t) := \gamma(t) - \delta(t)f'(\gamma(t))$$

for all $t \in {}^\star[0,1]$.

Since $\gamma_\delta \in {}^\star C([0,1], E)$,

$$\delta_{\frac{d}{2}}(0) = \delta_{\frac{d}{2}}(1) = 0 \quad \wedge \quad u(0) = u(1) = 0$$

and

$$\gamma_\delta(0) = \gamma(0) - \delta(0)f'(\gamma(0)) = x_1 \quad \wedge \quad \gamma_\delta(1) = \gamma(1) - \delta(1)f'(\gamma(1)) = x_2,$$

it follows that $\gamma_\delta \in {}^\star\Gamma$. Clearly, $\gamma_\delta({}^\star[0,1]) \subseteq ns({}^\star E)$.

We claim that

$$\forall t \in {}^\star[0,1] \quad f(\gamma_\delta(t)) \ < \ k_1.$$

This is certainly true for $t \in W$, because there $\delta(t) = \xi(t)$, ξ verifies (B.6) and, as $t \notin U$, $f(\gamma(t)) < k_1$; it is also true for $t \in V \setminus U$ because, when this is the case, $f'(\gamma(t)) \not\approx 0$ and $f(\gamma(t)) < k_1$ therefore, since $\delta(t) \approx 0$, by Lemma 5.19,

$$\forall t \in V \setminus U \quad f(\gamma_\delta(t)) \ \leq \ f(\gamma(t)) \ < \ k_1.$$

Now take $t \in U$; in this case, $\delta(t) = b$, then, by Lemma 5.19 and the definition of b,

$$f(\gamma_\delta(t)) = f(\gamma(t) - bf'(\gamma(t))) < f(\gamma(t)) - b\frac{\| f'(\gamma(t)) \|^2}{2} < k_1.$$

Hence

$$\forall t \in {}^\star[0,1] \quad f(\gamma_\delta(t)) \ < \ k_1,$$

which contradicts the definition of k_1 and therefore d must be infinitesimal. Consequently, k_1 is a critical value of f. ■

B.0.2 A discrete proof

Suppose $n \in \mathbb{N}$, $n \geq 2$, $E = \mathbb{R}^n$ and

$$\{e_1, e_2, \cdots, e_n\}$$

is the usual basis of $\mathbb{R}^n$; if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, ∂f_u will denote the directional derivative along the vector u, so that $\frac{\partial f}{\partial x_k} = \partial f_{e_k}$ $(k = 1, 2, \cdots, n)$.

As usual, B^A denotes the set of all mappings from the set A into the set B.

Lemma B.3 [BMN+05] *Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and define*

$$\begin{aligned} b : \mathbb{R}^n &\to \{1, 2, \cdots, n\} \\ x &\mapsto b(x) := \max\left\{k = 1, 2, \cdots, n \;:\; \left|\tfrac{\partial f}{\partial x_k}(x)\right| = \max\left\{\left|\tfrac{\partial f}{\partial x_i}(x)\right| \;:\; i = 1, 2, \cdots, n\right\}\right\}. \end{aligned}$$

If $a \in {}^\star\mathbb{R}^n$ is such that $f'(a) \not\approx 0$, then $\frac{\partial f}{\partial x_{b(a)}}(a) \not\approx 0$. Furthermore, if

$$u_{b(a)} = \begin{cases} -e_{b(a)} & \text{if } \frac{\partial f}{\partial x_{b(a)}}(a) > 0 \\ e_{b(a)} & \text{if } \frac{\partial f}{\partial x_{b(a)}}(a) < 0 \end{cases}$$

then, $0 \not\approx \partial f_{u_{b(a)}}(a) < 0$.

Proof. Suppose that $a \in {}^\star\mathbb{R}^n$ is such that $f'(a) \not\approx 0$. Since

$$f'(a) \not\approx 0 \Leftrightarrow \| f'(a) \| \not\approx 0 \Leftrightarrow \sqrt{\left(\frac{\partial f}{\partial x_1}(a)\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}(a)\right)^2} \not\approx 0,$$

there exists $i = 1, 2, \cdots, n$ such that $\frac{\partial f}{\partial x_i}(a) \not\approx 0$ and, consequently, $\frac{\partial f}{\partial x_{b(a)}}(a) \not\approx 0$. ■

The following is easily provable analogously to Lemma 5.19.

Lemma B.4 *Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and $a \in ns({}^\star\mathbb{R}^n)$ be such that $0 \not\approx \partial f_u(a) < 0$, where u is a fixed unitary vector in ${}^\star\mathbb{R}^n$. Then,*

$$\forall\ 0 < h \approx 0 \quad f(a + hu) < f(a).$$

In order to simplify the proof of Theorem B.1, we will prove the following equivalent version:

Theorem B.5 *Let E be a finite dimensional real Banach space and $f \in C^1(E,\mathbb{R})$. Suppose that*

1. *there exists $e \in E$ and $s \in \mathbb{R}^+$ such that $\| e \| > s$ and*

$$k_0 := \max\{f(0), f(e)\} \;<\; \min_{\|x\|=s} f(x);$$

2. *$\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0 \;\wedge\; \gamma(1) = e\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;*

3. *f is coercive.*

Then k_1 is a critical value of f.

Proof. [BMN+05] Since f is coercive, there exists $r \in \mathbb{R}^+$ such that $\| e \| < r$ and if $\| x \| > r$ then $f(x) > k_1 + 1$, therefore we may assume without loss of generality that

$$\forall \gamma \in \Gamma \quad \gamma([0,1]) \;\subseteq\; [-r,r]^n; \tag{B.7}$$

fix such an r, pick $N \in \mathbb{N}$ such that $N > r+1$, $M \in {}^\star\mathbb{N}_\infty$, define

$$h \;:=\; \frac{1}{M} \approx 0$$

and make

$$\mathcal{R} \;:=\; {}^\star]-N,N[^n \cap h{}^\star\mathbb{Z}^n \;=\; \Big\{a_i := -N + ih \;:\; 0 < i < 2NM \;\wedge\; i \in {}^\star\mathbb{N}\Big\}^n. \tag{B.8}$$

$\mathcal{R}$ is a hyperfinite subset of ${}^\star\mathbb{R}^n$ and elements $x_\sigma \in \mathcal{R}$ are of the form (see (B.8) above)

$$\begin{aligned} x_\sigma &:= (x_{\sigma(1)}, \cdots, x_{\sigma(n)}) \\ &= (-N + \sigma(1)h, \cdots, -N + \sigma(n)h) \end{aligned}$$

where $\sigma \in \{1,\cdots,2MN-1\}^{\{1,\cdots,n\}}$.

If $a = (a_1,\cdots,a_n)$ and $b = (b_1,\cdots,b_n)$ are two elements of $\mathbb{R}^n$ such that for all $i \in \{1,\cdots,n\}$, $a_i < b_i$, then we write

$$[a,b[\;:=\; \Pi_{i=1}^n [a_i,b_i[\;=\; [a_1,b_1[\times \cdots \times [a_n,b_n[.$$

Let us introduce a bit more of notation:

$$\overrightarrow{t} \;:=\; (t,\cdots,t) \qquad (t \in {}^\star\mathbb{R}),$$

so that, with some obvious abuse, for some $x_{\sigma_e} \in \mathcal{R}$

$$0 \in [0, \overrightarrow{h}[\quad \wedge \quad e \in [x_{\sigma_e}, x_{\sigma_e} + \overrightarrow{h}[.$$

Define

$$\begin{aligned} \mathcal{P} \;\; := \;\; & \Big\{ p({}^\star\mathbb{N}) \;:\; p \in {}^\star \Big((\mathbb{R}^n)^{\mathbb{N}} \Big) \;\wedge\; p({}^\star\mathbb{N}) \subseteq \mathcal{R} \;\wedge\; p(1) = 0 \\ & \wedge \Big[\exists \omega \in {}^\star\mathbb{N}\; p(\omega) = x_{\sigma_e} \;\wedge\; \forall n \in {}^\star\mathbb{N}\; [n \geq \omega \Rightarrow p(n) = x_{\sigma_e}] \Big] \\ & \wedge \Big[\; \forall i \in {}^\star\mathbb{N} \parallel p(i) - p(i+1) \parallel \leq \; \sqrt{n} h \Big] \Big\}. \end{aligned}$$

Claim 1: *$\mathcal{P}$ is a hyperfinite set and*

$$c \;:=\; \min_{p({}^\star\mathbb{N}) \in \mathcal{P}} \; \max_{x \in p({}^\star\mathbb{N})} f(x) \tag{B.9}$$

is well defined.

Note that $\mathcal{R}$ is hyperfinite, say it has $card(\mathcal{R})$ elements; each $p({}^\star\mathbb{N})$ is an internal subset of $\mathcal{R}$, hence also hyperfinite, and

$$\max_{x \in p({}^\star\mathbb{N})} f(x)$$

is well defined; moreover, $\mathcal{P}$ is an internal set of internal parts of $\mathcal{R}$ and there cannot be more than $2^{card(\mathcal{R})}$ of these, so that $\mathcal{P}$ is itself hyperfinite and c is indeed well defined.

Claim 2: $c \approx k_1$.

Let us see first that

$$k_1 \;<\; c \quad \vee \quad c \;\approx\; k_1. \tag{B.10}$$

Let $q({}^\star\mathbb{N}) \in \mathcal{P}$ be such that $c = \max_{x \in q({}^\star\mathbb{N})} f(x)$; suppose that $\omega \in {}^\star\mathbb{N}$ is such that

$$\forall i \in {}^\star\mathbb{N} \; [\; i \geq \omega - 1 \;\Rightarrow\; q(i) = x_{\sigma_e} \;],$$

fill in linearly between each $q(i)$ and $q(i+1)$ and between x_{σ_e} and e, and reparametrize in ${}^\star[0,1]$, i.e., define

$$\gamma(t) \;:=\; \begin{cases} q(i) + \omega \left(t - \frac{i}{\omega}\right) \big(q(i+1) - q(i)\big) & \text{if} \quad \frac{i}{\omega} \leq t \leq \frac{i+1}{\omega} \;\wedge\; 0 \leq i \leq \omega - 2 \\ x_{\sigma_e} + \omega \left(t - \frac{\omega-1}{\omega}\right) (e - x_{\sigma_e}) & \text{if} \quad \frac{\omega-1}{\omega} \leq t \leq 1 \end{cases} .$$

Note that $\gamma \in {}^\star\Gamma$ and

$$k_1 \quad \leq \max_{t \in {}^\star[0,1]} f(\gamma(t)) \; \approx \; \max_{x \in q({}^\star\mathbb{N})} f(x) \; = \; c,$$

so that (B.10) is proven.

Now take $\varepsilon \in \,]0,1],\ \gamma \in \Gamma$ such that

$$k_1 \quad \leq \quad \max_{t \in [0,1]} f(\gamma(t)) < k_1 + \varepsilon$$

and recall from (B.7) that

$$\gamma([0,1]) \; \subseteq \; [-r,r]^n.$$

Pick $\Omega \in {}^\star\mathbb{N}_\infty$ such that

$$\left\| \gamma\left(\frac{i+1}{\Omega}\right) - \gamma\left(\frac{i}{\Omega}\right) \right\| \; < \; h \qquad (0 \leq i \leq \Omega - 1). \tag{B.11}$$

With the points $\gamma\left(\frac{i}{\Omega}\right)$, $i = 0, 1, \cdots, \Omega$, we will define a function $p : {}^\star\mathbb{N} \to {}^\star\mathbb{R}^n$ in the following way: for $i \in \{1, 2, \cdots, \Omega - 1\}$, we choose $p(i) \in \mathcal{R}$ such that

$$\gamma\left(\frac{i-1}{\Omega}\right) \; \in \; [p(i), p(i) + \overrightarrow{h}\,[$$

and

$$p(i) = x_{\sigma_e} \quad \text{for all } i \geq \Omega.$$

Observe that indeed

$$\|p(i+1) - p(i)\| \; \leq \; \sqrt{n}h.$$

It is easily seen from (B.11) and the continuity of f, that

$$c \; \leq \; \max_{x \in p({}^\star\mathbb{N})} f(x) \; \approx \; \max_{t \in {}^\star[0,1]} f(\gamma(t)) \; < \; k_1 + \varepsilon$$

so that

$$c \; < \; k_1 + \varepsilon;$$

this together with (B.10) implies our claim that $c \approx k_1$.

Note that

$$\forall p \in {}^\star\left((\mathbb{R}^n)^{\mathbb{N}}\right) \quad \left[\max_{x \in p({}^\star\mathbb{N})} f(x) \approx k_1 \quad \Rightarrow \; p({}^\star\mathbb{N}) \subseteq {}^\star[-r,r]^n \right].$$

Claim 3: *$f^{-1}\Big|_{\mathcal{R}}(c)$ contains at least one almost critical point, that is, there exists $a \in \mathcal{R}$ such that $f(a) = c$ and $f'(a) \approx 0$.*

Suppose this is not the case, that is,

$$\forall x \in \mathcal{R} \quad \big[f(x) = c \;\Rightarrow\; f'(x) \not\approx 0\big]. \tag{B.12}$$

Let $p_{min}(^\star\mathbb{N}) \in \mathcal{P}$ be such that

$$\max_{x \in p_{min}(^\star\mathbb{N})} f(x) = \min_{p(^\star\mathbb{N}) \in \mathcal{P}} \; \max_{x \in p(^\star\mathbb{N})} f(x) = c.$$

Note that

$$[x \in p_{min}(^\star\mathbb{N}) \;\wedge\; f(x) = c] \quad \Rightarrow \quad [x \not\approx 0 \;\wedge\; x \not\approx e]$$

and let

$$\nu \;:=\; \max\{i \in {}^\star\mathbb{N} \;:\; f(p_{min}(i)) = c\}.$$

Let us first do an internal partition of $f^{-1}(c) \cap p_{min}(^\star\mathbb{N})$. Define recursively

$$\begin{aligned} i_1 \;&:=\; \min\{i \in {}^\star\mathbb{N} \;:\; f(p_{min}(i)) = c\} \\ \overline{i_1} \;&:=\; \max\{j \in {}^\star\mathbb{N} \;:\; \forall i \in {}^\star\mathbb{N}\; [i_1 \le i \le j \Rightarrow f(p_{min}(i)) = c]\} \end{aligned}$$

$$\begin{aligned} i_{\ell+1} \;&:=\; \min\{i \in {}^\star\mathbb{N} \;:\; \overline{i_\ell} < i \;\wedge\; f(p_{min}(i)) = c\} \\ \overline{i_{\ell+1}} \;&:=\; \max\{j \in {}^\star\mathbb{N} \;:\; \forall i \in {}^\star\mathbb{N}\; [i_{\ell+1} \le i \le j \Rightarrow f(p_{min}(i)) = c]\}. \end{aligned}$$

Let $\omega \in {}^\star\mathbb{N}$ be such that

$$\forall i \in {}^\star\mathbb{N}\; [\; i \ge \omega \;\Rightarrow\; p_{min}(i) = x_{\sigma_e} \;].$$

For some $\kappa \in {}^\star\mathbb{N}$,

$$1 \;< i_1 \;\le\; \overline{i_1} \;<\; \cdots \;< i_\kappa \;\le\; \overline{i_\kappa} \;=\; \nu \;<\; \omega.$$

With

$$C_j \;:=\; \{p_{min}(i) \;:\; i_j \le i \le \overline{i_j}\} \qquad (1 \le j \le \kappa),$$

we have

$$f^{-1}(c) \cap p_{min}(^\star\mathbb{N}) \;= \dot{\bigcup}_{1 \le j \le \kappa} C_j \;:=\; \mathcal{C},$$

that is, $f^{-1}(c) \cap p_{min}({}^{\star}\mathbb{N})$ is the disjoint union of the C_j.

For each $1 \leq j \leq \kappa$, repartition the C_j the following way using the function b from Lemma B.3:

$$\begin{aligned} m_{j,1} &:= \max\left\{m \in {}^{\star}\mathbb{N} \ : \ \forall i \in {}^{\star}\mathbb{N} \ [\ i_j \leq i \leq m \Rightarrow b(p_{min}(i)) = b(p_{min}(i_j))\ ]\right\} \\ m_{j,\ell+1} &:= \max\left\{m \in {}^{\star}\mathbb{N} \ : \ \forall i \in {}^{\star}\mathbb{N} \ [\ m_{j,\ell} < i \leq m \Rightarrow b(p_{min}(i)) = b(p_{min}(m_{j,\ell}+1))\ ]\right\} \end{aligned}$$

$$\begin{aligned} C_{j,1} &:= \left\{p_{min}(i) \ : \ i_j \leq i \leq m_{j,1}\right\} \\ C_{j,\ell} &:= \left\{p_{min}(i) \ : \ m_{j,\ell-1} < i \leq m_{j,\ell}\right\} \quad (\ell > 1). \end{aligned}$$

For some specific sequence $(\chi_j)_{1 \leq j \leq \kappa}$,

$$1 < i_1 \leq m_{1,1} < \cdots < m_{1,\chi_1} = \overline{i_1} < \cdots < i_\kappa \leq m_{\kappa,1} < \cdots < m_{\kappa,\chi_\kappa} = \overline{i_\kappa} = \nu < \omega,$$

$$C_j = \dot{\bigcup}_{1 \leq \ell \leq \chi_j} C_{j,\ell}$$

and

$$f^{-1}(c) \cap p_{min}({}^{\star}\mathbb{N}) = \dot{\bigcup}_{1 \leq j \leq \kappa} \dot{\bigcup}_{1 \leq \ell \leq \chi_j} C_{j,\ell}.$$

Observe that $C_{j,\ell}$ were built so that the function b is constant on each of them. We will see that under these assumptions a function $p^{-}({}^{\star}\mathbb{N}) \in \mathcal{P}$ for which $\max f(p^{-}({}^{\star}\mathbb{N})) < c$ may be built, thus contradicting the definition of c, so that condition (B.12) cannot hold and Claim 3 will be proven.

The procedure consists of two instances:

1. To define a convenient *multi-valued internal* — at most 1-2 — function $P : \{1, \cdots, \omega\} \to \mathcal{R}$

   (a) P outside $\mathcal{C}$:

$$P(i) := \begin{cases} p_{min}(i) & p_{min}(i) \notin \mathcal{C} \\ p_{min}(i_j - 1) + hu_{b(p_{min}(i_j))} & i = i_j - 1 \\ p_{min}(\overline{i_j} + 1) + hu_{b(p_{min}(\overline{i_j}))} & i = \overline{i_j} + 1 \\ & (1 \leq j \leq \kappa) \end{cases}$$

P is 1-2 at the $i_j - 1$ and $\overline{i_j} + 1$.

(b) P inside $\mathcal{C}$:

i. **First step**

$$P(i) \;:=\; p_{min}(i) + hu_{b(p_{min}(i))} \quad \text{if} \quad p_{min}(i) \in \mathcal{C} \;\; (1 \leq j \leq \kappa).$$

Observe that the distances between consecutive points outside $\mathcal{C}$ are simply not changed because the same happens with the points themselves; note also that *adding one point before the begin and after the end* of each C_j as we did in (a) keeps the passage "into" and "out of" $\mathcal{C}$ within the bound $\sqrt{n}h$.

Also observe that, by Lemma B.4 and condition (B.12), when $1 \leq j \leq \kappa$,

$$\begin{aligned} f(p_{min}(i) + hu_{b(p_{min}(i))}) &< c \qquad \text{if} \qquad p_{min}(i) \in C_j \\ f(p_{min}(i)) &< c \qquad \text{if} \qquad p_{min}(i) \notin \mathcal{C} \\ f(p_{min}(i_j - 1) + hu_{b(p_{min}(i_j))}) &< f(p_{min}(i_j - 1)) < c \\ f(p_{min}(\overline{i_j} + 1) + hu_{b(p_{min}(\overline{i_j}))}) &< f(p_{min}(\overline{i_j} + 1)) < c. \end{aligned}$$

ii. **Second step**.

We must take care of passages from $C_{j,\ell}$ to $C_{j,\ell+1}$.

Let $p_{min}(i) := \alpha$ be the last element of $C_{j,\ell}$ and $p_{min}(i+1) := \beta$ the first of $C_{j,\ell+1}$; it is easily seen that $u_{b(\alpha)} \perp u_{b(\beta)}$ because the relevant partial derivatives are S-continuous and non-infinitesimal at α and at β; as their distance is at most $\sqrt{n}h$, α and β are vertices of an interval of the grid $\mathcal{R}$, say I. Define possibly one more image for one or both of these $P(\cdot)$ by:

$$\begin{cases} P(i) = \alpha + hu_{b(\beta)} & \text{if } u_{b(\beta)} \text{ points to the outside of } I \\ P(i+1) = \beta + hu_{b(\alpha)} & \text{if } u_{b(\alpha)} \text{ points to the outside of } I \end{cases}.$$

Note that, by Lemma B.4, α and β cannot be consecutive vertices when $u_{b(\beta)} = \frac{\beta - \alpha}{\|\beta - \alpha\|}$ or $u_{b(\alpha)} = \frac{\alpha - \beta}{\|\alpha - \beta\|}$.

Again

$$f(P(j)) \quad < \quad c \qquad\qquad (j = i, i+1)$$

because of Lemma B.4, the fact that f' is S-continuous and (B.12).

2. By means of adequate shifts in i, a function p^- may be built from P for which $\max f(p^-({}^\star\mathbb{N})) < c$, an impossibility as mentioned above.

Claim 4: *k_1 is a critical value of f.*

This follows from Claim 2, Claim 3 and the continuity of f and f'. ∎

Bibliography

[ACH97] Leif O.(ed.) Arkeryd, Nigel J.(ed.) Cutland, and C.Ward (ed.) Henson. *Nonstandard analysis: theory and applications. Proceedings of the NATO Advanced Study Institute on nonstandard analysis and its applications, Edinburgh, UK, June 30–July 13, 1996.* NATO ASI Series. Series C. Mathematical and Physical Sciences. 493. Dordrecht: Kluwer Academic Publishers. xiv, 366 p. , 1997. {1}

[AE84] Jean-Pierre Aubin and Ivar Ekeland. *Applied nonlinear analysis.* Pure and Applied Mathematics. A Wiley-Interscience Publication. New York etc.: John Wiley & Sons. XI, 518 p. , 1984. {54}

[AHKFL86] Sergio Albeverio, Raphael Høegh-Krohn, Jens Erik Fenstad, and Tom Lindstrøm. *Nonstandard methods in stochastic analysis and mathematical physics.* Pure and Applied Mathematics, Vol. 122. Orlando etc.: Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers). XI, 514 p., 1986. {12,26,28}

[And82] Robert M. Anderson. Star-finite representations of measure spaces. *Trans. Am. Math. Soc.*, 271:667–687, 1982. {26,29,31,34}

[AR73] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.*, 14:349–381, 1973. {4,70}

[BCN80] Haïm Brézis, Jean-Michel Coron, and Louis Nirenberg. Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. *Commun. Pure Appl. Math.*, 33:667–684, 1980. {2,4,60,72}

[BMN+05] Maria João Borges, Natália Martins, Vítor Neves, João Teixeira, and Manfred Wolff. Mountain pass theorems without the deformation lemma. i. finite dimensional spaces. Technical report, Cadernos de Matemática CM05/I-36, Departamento de Matemática da Universidade de Aveiro, 2005. {76,98,99,101,102}

[CL55] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations.* New York, Toronto, London: McGill-Hill Book Company, Inc. XII, 429 p., 1955. {40,94}

[CLW90] L. Caklovic, Shujie Li, and M. Willem. A note on Palais-Smale condition and coercivity. *Differ. Integral Equ.*, 3(4):799–800, 1990. {70}

[CNOSP95] Nigel J.(ed.) Cutland, Vítor (ed.) Neves, Franco (ed.) Oliveira, and José (ed.) Sousa-Pinto. *Developments in nonstandard mathematics. Proceedings of the international colloquium on nonstandard mathematics, held at the University of Aveiro, Portugal, July 18-22, 1994.* Pitman Research Notes in Mathematics Series. 336. Harlow: Longman. New York, NY: Wiley, 260 p., 1995. {1}

[Coh80] Donald L. Cohn. *Measure Theory.* Birkhäuser Boston 373 p. , 1980. {38}

[Cor99] Jean-Noël Corvellec. Quantitative deformation theorems and critical point theory. *Pac. J. Math.*, 187(2):263–279, 1999. {4}

[Cou50] Richard Courant. *Dirichlet's principle, conformal mapping and minimal surfaces.* (Pure and Applied Mathematics. A Series of Texts and Monographs. III) New York: Interscience Publishers, Inc. XIII, 330 p. , 1950. {78}

[Cut83] Nigel J. Cutland. Nonstandard measure theory and its applications. *Bull. Lond. Math. Soc.*, 15:529–589, 1983. {26}

[Cut95] Nigel Cutland. Loeb measure theory. In *Cutland, Nigel J. (ed.) et al., Developments in nonstandard mathematics. Proceedings of the international colloquium on nonstandard mathematics, held at the University of Aveiro, Portugal, July 18-22, 1994. Harlow: Longman. Pitman Res. Notes Math. Ser. 336, 151-177* . 1995. {26,28,30,31,33,34}

[Cut97] Nigel J. Cutland. Nonstandard real analysis. In *Arkeryd, Leif O. (ed.) et al., Nonstandard analysis: theory and applications. Proceedings of the NATO Advanced Study Institute on nonstandard analysis and its applications, Edinburgh,* *UK, June 30–July 13, 1996. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 493, 51-76* . 1997. {12,23}

[Cut00] Nigel J. Cutland. *Loeb measures in practice: Recent advances. EMS lectures 1997.* Lecture Notes in Mathematics. 1751. Berlin: Springer. xi, 122 p. , 2000. {26,28,29,96}

[Dav77] Martin Davis. *Applied Nonstandard Analysis.* John Wiley & Sons 181 p. , 1977. {20}

[Eke79] Ivar Ekeland. Nonconvex minimization problems. *Bull. Am. Math. Soc., New Ser.*, 1:443–474, 1979. {2,66,67}

[Fig88] Djairo G. de Figueiredo. Lectures on the ekeland variational principle with applications and detours. Technical report, International Centre for Theoretical Physics, Trieste, Italy, 1988. {66}

[GT01] Maria do Rosário Grossinho and Stepan Agop Tersian. *An introduction to minimax theorems and their applications to differential equations.* Nonconvex Optimization and Its Applications. 52. Dordrecht: Kluwer Academic Publishers. xii, 269 p. , 2001. {5,7,60,66,71,81}

[Hen97] C.Ward Henson. Foundations of nonstandard analysis. A gentle introduction to nonstandard extensions. In *Arkeryd, Leif O. (ed.) et al., Nonstandard analysis: theory and applications. Proceedings of the NATO Advanced Study Institute on nonstandard analysis and its applications, Edinburgh, UK, June 30–July 13, 1996. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 493, 1-49* . 1997. {12}

[HL85] Albert E. Hurd and Peter A. Loeb. *An introduction to nonstandard real analysis.* Pure and Applied Mathematics, 118. Orlando etc.: Academic Press, Inc. (Harcourt Brace Javanovich, Publishers). XII; 232 p., 1985. {12,14,17,19,20,22}

[Jab03] Youssef Jabri. *The mountain pass theorem. Variants, generalizations and some applications.* Encyclopedia of Mathematics and Its Applications 95. Cambridge: Cambridge University Press. xii, 369 p. , 2003. {5,60,66}

[JLJ98] Jürgen Jost and Xianqing Li-Jost. *Calculus of variations.* Cambridge Studies in Advanced Mathematics. 64. Cambridge: Cambridge University Press. xvi, 323 p., 1998. {78}

[KR04] V. Kanovei and Michael Reeken. *Nonstandard analysis, axiomatically.* Springer Monographs in Mathematics. Berlin: Springer. xvi, 408 p., 2004. {12}

[Lan93] Serge Lang. *Real and functional analysis. 3. ed.* Graduate Texts in Mathematics. 142. New York: Springer-Verlag. xiv, 580 p., 1993. {57}

[Lin88] Tom Lindstrøm. An invitation to nonstandard analysis. In *Nonstandard analysis and its applications, Pap. Conf., Hull/UK 1986, Lond. Math. Soc. Stud. Texts 10, 1-105* . 1988. {12,15,18,20,26,29,31}

[Loe75] P. A. Loeb. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Transactions of the American Mathematical Society*, 211:113–122, 1975. {26}

[Loe97] Peter A. Loeb. Nonstandard analysis and topology. In *Arkeryd, Leif O. (ed.) et al., Nonstandard analysis: theory and applications. Proceedings of the NATO Advanced Study Institute on nonstandard analysis and its applications, Edinburgh, UK, June 30–July 13, 1996. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 493, 77-89* . 1997. {12}

[LS34] L. Lusternik and L. Schnirelmann. *Méthodes topologiques dans les problèmes variationnels. I. Pt. Espaces à un nombre fini de dimensions. Traduit du russe par J. Kravtchenko.* Paris: Hermann & Cie. 51 S., 5 Fig. , 1934. {3,66}

[Mar97] Natália Martins. Introdução à Análise Não-Standard e sua Aplicação à Medida e Integral de Loeb. Master's thesis, Universidade de Aveiro, 1997. {12,26,28,29,34,35}

[Mar05] Natália Martins. Nonstandard proofs of some critical point theorems, in preprints of the third junior european meeting on control, optimization, and computation. Technical report, Cadernos de Matemática CM05/I-04, Departamento de Matemática da Universidade de Aveiro, 2005. {81,83,88,89}

[Maw02] J. Mawhin. Critical point theory and applications to nonlinear differential equations. Technical report, University of Utah, (http://www.math.utah.edu/ rudd/minicourse/mawhin.pdf), 2002. {66,67,72}

[MM02] Jorge J. Moré and Todd S. Munson. Computing mountain passes. Technical report, Argonne National Laboratory, Preprint ANL/MCS-P957-0522, (htpp://www-unix.mcs.anl.gov/ more//papers/saddle.pdf), 2002. {69}

[MNa] Natália Martins and Vítor Neves. Nonstandard Discrete derivatives and Existence Theorems for ODEs. *Rediconti del Seminário Matemático dell'Università e del Politecnico di Torino*, Special Issue "Control Theory and Stabilization I", Vol. 63, 4, 383-395, 2005. {38,40,93}

[MNb] Natália Martins and Vítor Neves. Nonstandard Palais-Smale conditions. *Preprints of the Conference on Non Standard Mathematics NSM2004, Department of Mathematics, University of Aveiro, July 5-10, 2004* - approved for publication in January 2006. {49,50,51,63,64,72}

[Nev01] Vítor Neves. Análise não-standard de espaços localmente convexos. Technical report, Departamento de Matemática da Universidade de Aveiro, (http://www.mat.ua.pt/vneves/nsa/elc.pdf), 2001. {14,15,18}

[PS64] R.S. Palais and S. Smale. A generalized Morse theory. *Bull. Am. Math. Soc.*, 70:165–172, 1964. {2,45}

[PS84] Patrizia Pucci and James Serrin. Extensions of the mountain pass theorem. *J. Funct. Anal.*, 59:185–210, 1984. {5}

[Rab86] Paul H. Rabinowitz. Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conference, 65, AMS, Providence, Rhode Island*, 1986. {66,71}

[Rao87] M. M. Rao. *Measure Theory and Integration.* A Wiley-Interscience Publication, John Wiley & Sons 540 p. , 1987. {38,39}

[Rob61] Abraham Robinson. Non-standard analysis. *Nederl. Akad. Wet., Proc., Ser. A*, 64:432–440, 1961. {1}

[Ros97] David A. Ross. Loeb measure and probability. In *Arkeryd, Leif O. (ed.) et al., Nonstandard analysis: theory and applications. Proceedings of the NATO Advanced Study Institute on nonstandard analysis and its applications, Edinburgh, UK, June 30–July 13, 1996. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 493, 91-120* . 1997. {26}

[RTK98] Nadezhda Ribarska, Tsvetomir Tsachev, and Mikhail Krastanov. The intrinsic mountain pass principle. *Topol. Methods Nonlinear Anal.*, 12(2):309–322, 1998. {4}

[San93] Luís Sanchez. *Métodos da teoria de pontos críticos.* Textos de Matemática. 1. Lisboa: Universidade de Lisboa, Faculdade de Ciências, Departamento de Matemática. 159 p. , 1993. {2}

[SB86] K.D. Stroyan and José Manuel Bayod. *Foundations of infinitesimal stochastic analysis.* Studies in Logic and the Foundations of Mathematics, Vol. 119. Amsterdam- New York-Oxford: North-Holland. XII, 478 p. , 1986. {26,28,35}

[SL76] K.D. Stroyan and W.A.J. Luxemburg. *Introduction to the theory of infinitesimals.* Pure and Applied Mathematics. 72. New York-San Francisco-London: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers. XIII, 326 p. , 1976. {12,25}

[Str78] K.D. Stroyan. Infinitesimal calculus on locally convex spaces. I: Fundamentals. *Trans. Am. Math. Soc.*, 240:363–383, 1978. {26}

[Tuc93] Curtis Tuckey. *Nonstandard methods in the calculus of variations.* Pitman Research Notes in Mathematics Series. 297. Harlow: Longman Scientific & Technical. New York: John Wiley & Sons, Inc. xiii, 94 p. , 1993. {24,35}

[Wei] Karl Weierstrass. *Über das sogenannte Dirichlet'sche Princip. (Gelesen in der Königl. Akad. Wiss. 14 Juli 1870).* Mathematische Werke von Karl Weierstrass. 2. Band, Abhandlungen II. Berlin, Mayer& Müller 1895. {2}

[Wil83] Michel Willem. *Lectures on critical point theory. (Given at the University of Brasilia in January and February 1983).* Louvain-la-Neuve, Belgique: Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée. 61 p., 1983. {3,66}

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